

A N

INTRODUCTION

R

E. Dono L. Evans
TO THE

to John Black

DOCTRINE

OF

FLUXIONS.

Revised by several Gentlemen well skill'd in
the MATHEMATICS.

— *Felicibus inde*
Ingeniis aperitur Iter. — CLAUDIAN.

L O N D O N :

Printed by E. OWEN in *Hand-Court, Holbern*;
And Sold by JOHN NOON in the *Poultry*.

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T H E
P R E F A C E.

Priming

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 *S the Want of a plain and easy Introduction, whereby the industrious Learner, without the Assistance of a Master, might arrive to a small Degree of Skill in the Doctrine of Fluxions, was the only Inducement to this Work; 'tis hoped the following Sheets will need no Apology; they being principally designed to supply this Deficiency, and to assist the ingenious Student, by removing, in some Measure, those Clouds of Darkness and Obscurity, in which the Treatises on this Subject are too often involved.*

Tho'

Tho' in what is here delivered, neither any of the most difficult and abstruse Parts of the Doctrine of Fluxions are contained, nor many new Discoveries made; yet as it may tend to give the Learner a perfect Notion of, and excite him to search farther into that noble and useful Science, it may justly deserve some Favour from the Public.

The Entrance on every Science is always the most difficult; and the gaining clear and just Ideas at first, as it is the most uneasy Part, so it is the surest Pledge of future Success.

The Mind cannot be easily brought to conceive of things entirely unknown before, or in a Manner different from that to which it has hitherto been accustomed. Men are used to talk and reason of finite Quantities only; but here we must conceive of those which are less than can be assigned; of those which are but just coming into Being, or just vanishing out of it.

*Here Quantities are considered as generated by a continual Increase, after the Manner of a Space, which a Thing or Point in Motion describes: Thus a Line is conceived to be generated by a Point in Motion, a Superficies by a Line,
and*

The P R E F A C E. v

and a Solid by a Superficies. Now as a Line generated by a Point must be passed over, or described, with some certain Degree of Velocity in the describing Point, in every where or Point of that Line; so any Quantity may be conceived to flow, or increase, with a certain Degree of Velocity in every Point of its Description: And if this Velocity be not Uniform, but is either accelerated or retarded, then there will be a certain Degree of Increase, peculiar to every Point of the Thing described. Now the Velocity with which any Quantity flows at any particular Point, is what we call the Fluxion of that Quantity at that Point.

This Treatise being only designed as an Introduction, we have carefully avoided, in the Inverse Method, the giving of Examples where the Fluents of the Fluxions need Correction: We have also avoided, not only in the Inverse Method, but also in the Direct, all those laborious Operations which are more troublesome than instructive: And have endeavour'd to steer between an obscure Brevity on the one Hand, and a tedious Prolixity on the other.

The Questions left unanswered for the Amusement of the Learner, may be easily solved by an Application of the Rules here delivered.

The

The Examples for Illustration, are many of them borrowed from Books already wrote on this Subject, and have only to boast, that they are set here in a different, and perhaps, a clearer Light. The Whole is calculated for the unassisted Learner; and, we flatter ourselves, is a sufficient Guide to lead him to the Understanding of a compcat, tho' puzzling System: And if it answers this End, 'tis all that is either desired or expected.

EXETER,
April 30, 1751.

*This is the 1st Edition of
John Rowe's Fluxions.
The 2^d. came out with his
Name in 1757 with Alterations
and Additions. And the 3^d.
came out in 1767 much improved
with 13 Copper-Plates. I am
not aware that any other Edition
was printed. L. E. R. M. H.*



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E R R A T A

PAGE 4. *l.* 11. *dele* Moments or. *l.* 17. *l.* 4. *f.* now together, read be generated in the same Time. *l.* 9. *r.* certain constant ratio. *P.* 13. *l.* 3. *f.* at the Point D, or increases. *P.* 14. *l.* 9. *r.* xy^2x' . *P.* 22. *l.* 2. *dele*, at the Point A. *l.* 3. *dele* at that Point. *l.* ult. *r.* $m \cdot a - x^n = n \cdot a - x^{n-1}x$. *P.* 28. *l.* 9. *f.* =, *r.* out. *P.* 33. *l.* 1. after *f.*ies add, Hence $cx^2 + \frac{4a}{x}$ = the whole inside Superficies. *P.* 40. *l.* 8. *f.* a , *r.* a^2 . *P.* 51. *l.* 5. *f.* p , *r.* a . *l.* ult. *r.* $\frac{xy}{y}$. *P.* 53. *l.* 9. *f.* ab^3 , *r.* ab^2 . *P.* 60. *l.* 16. *f.* EF (CE *r.* EF: (CE. *l.* 22. *f.* $b - x$ *r.* $\times b - x$. *P.* 62. *l.* 6. after Quantity, add more than Unity. *P.* 70. *l.* 20. *f.* $\frac{1}{10}$, *r.* $\frac{1}{10}$. *P.* 74. *l.* ult. *f.* $i = 0$, *r.* j are $= 0$). *P.* 75. *l.* 1. *f.* $-2xx$ *r.* $-2xx$. *l.* 2. *r.* $\frac{x - ab^2x - x^3x}{x^2b^2 - x^2}$ (see *Art.* 11.) *i.* *e.* *l.* 11. *r.* $b^2x - x^3$. *P.* 84. *l.* 15. *f.* and EHm, *r.* EHb. *P.* 87. *l.* 11. *f.* $ax^{\frac{1}{2}}$ *r.* $\overline{ax}^{\frac{1}{2}}$. *l.* 15. *f.* $\overline{ax}^3$ *r.* $\overline{ax}^{\frac{1}{2}}$. *P.* 88. *l.* ult. *f.* $m - 1$, *r.* $m - 1$. *P.* 94. *l.* 20. *r.* CB = y . *P.* 98. *l.* 8. *f.* $y \times x^2 \times y^2$ *r.* $y \times x^2 \times y^2$. *l.* 15. *f.* $y \times 1 + y^2$ *r.* $y \times 1 + y^2$. *P.* 101. *l.* ult. *f.* $a^2y^2 + b^4$, *r.* $a^2y^2 + b^4$. *P.* 107. *l.* 13 and 14. *f.* AD *r.* AE. *P.* 112. *l.* 6. *f.* $4x$, *r.* $4x^3$. *P.* 118. *l.* 13. *r.* $-\frac{5x^5}{128a^7}$. *l.* ult. *r.* $-\frac{x^4}{8a^3}$. *P.* 119. *l.* 11. *f.* $\frac{b}{a+b}$ *r.* $\frac{b}{a+x}$. *P.* 120. *l.* 13. *f.* $\frac{m-1}{2n}$ *r.* $\frac{m-n}{2n}$. *P.* 124. *l.* 15. *f.* 100 *r.* 99. *P.* 126. *l.* 5 and 6. *r.* and parallel. *P.* 128. *l.* 1. *r.* $\frac{4a^2x^2x^2 + 4x^4x^2}{a^4}$. *P.* 131. *l.* 1. *dele*, at the Point B. *P.* 132. *l.* 4. *r.* $\frac{1}{2}x^{\frac{1}{2}}x$. *P.* 134. *l.* 5. *f.* above, *r.* about. *P.* 135. *l.* 15. *dele* x . *P.* 137. *l.* 5. *f.* concave *r.* convex. *P.* 138. *l.* 1. *f.* $\times \frac{1}{2}b^2$ *r.* $\frac{1}{2}b^2$. *P.* 139. *l.* 10. *f.* Area *r.* Semicircumference. *P.* 142. *l.* 1. *f.* y *r.* y^3 . *P.* 143. *l.* 2. *f.* $\frac{bx^2}{a}$ *r.* $\frac{b^2x^2}{a^2}$. *l.* 6. *r.* $\frac{1}{2}cb^2a$. *P.* 144. *l.* 12. *dele*, of the Plane. *l.* penult. after is, add = the Segm. content = $2fax^2 - \frac{1}{2}fx^3$ wherefore. *P.* 147. *l.* 22. *f.* CN *r.* CK. *P.* 150. *l.* 8. *f.* md *r.* mD. *P.* 152. *l.* 13. *f.* $2ax$ *r.* $2a^2$. *l.* 14. *f.* $4ay - 4ax$, *r.* $4ax - 4ay$. *P.* 161. *l.* 3. *f.* a^2 , *r.* b^2 . and *f.* $2ab + b^2$, *r.* $\sqrt{2ab + b^2}$. *l.* 4. and 5. *f.* md *r.* ma. *l.* 6. *f.* a^2 *r.* b^2 .



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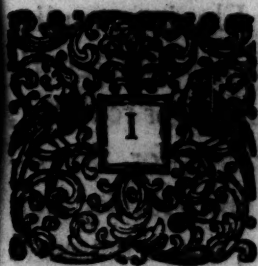
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AN
INTRODUCTION
TO THE
DOCTRINE
OF
FLUXIONS.



CHAP. I.



IN Order to render the *Doc-
trine of Fluxions* plain and
easy, we shall endeavour to
explain the Nature of FLUX-
ION in general, as deliver'd
both by Sir *Isaac Newton*
(the great Inventor of *Fluxions*) and by *Leib-
nitz* and his Followers.

B

DE-

DEFINITION I.

I. FLUXION (according to Sir *Isaac Newton*) is that whereby a Body in Motion is carried over a given Space in a given Time ; that is, *Fluxion* is the same as *Velocity*. Thus (Quantities being here conceived to be generated by continual Increase, after the Manner of a Space which a Thing or Point in Motion describes ; as a Line by a Point, a Superficies by a Line, and a Solid by a Superficies, in Motion :) if we imagine any Point A, to be mov'd along the Line A B the Velocity, or Swiftnefs, with which it moves at A B any Point of that Line is truly and properly the *Fluxion* of it at that Point. If this Velocity be uniform thro' the whole Line (*i. e.* if it has no Variation) the Fluxion will be every where the same ; but if it be any Way accelerated or retarded, then there will be a certain Degree of Velocity or Fluxion peculiar to every Point ; and the Velocity wherewith the first Velocity is either accelerated or retarded, is called the *Fluxion of the Fluxion*, or the *second Fluxion* : And again, if this Acceleration or Retardation be not uniform, but is increas'd in different Degrees of Proportion in different Parts of the Line ; then the Velocity of this Ac-

DOCTRINE of FLUXIONS.

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Acceleration or Retardation is called the *Third Fluxion*; and so on. This is the just and accurate Notion of FLUXION.

But, because in some Cases Velocity may not seem so particularly concern'd; it may be necessary to give the *Foreigners* Definition of it, that nothing be wanting to give the Learner as clear an Insight as possible into the Nature of the Subject; which, as it is highly necessary in all Parts of Science, so it is particularly in this, where an Obscurity of Conception in the Outset may lead him into endless Error and Confusion.

DEFINITION II.

II. Quantities are here suppos'd to be generated by a continual Increase, as before; and the indefinitely small Particles whereby they are continually increas'd, are call'd the *Fluxions* of those Quantities. Thus, in the Curve ABD , if bc be suppos'd to be indefinitely near and parallel to the Ordinate BC , and Be parallel to the Absciss AC ; then Be or Cc is call'd the *Fluxion* of the Absciss AC , and eb the *Fluxion* of the Ordinate CB ; and if nd be suppos'd indefinitely near and parallel to bc , and bm equal and parallel to cd or Be ; then the Difference between eb and mn is call'd the *Fluxion* of the *Fluxion* or the *Second Fluxion*; that is, the

DOCTRINE of FLUXIONS. 5

uniform Motions) it is evident that then the Increments would be accurately as the Fluxions or Velocities; since Velocity is always measured by the Space uniformly described in a given Time: But as in Curves, the Proportions of Increase or Decrease are continually varying, so it is evident there must be a different Degree of Fluxion or Velocity ascrib'd to every Point of these Increments; and that therefore these Increments are not exactly as the Fluxions. However, as the Point b is continually nearer to a Coincidence with the Tangent TBG the nearer it approaches the Point of Contact B ; so if we conceive the Ordinate cb to be moved on till it coincide with CB ; the very first Moment before its Coincidence, the Curve Bb , and Right-line BG will be infinitely, or rather indefinitely near to a Coincidence with each other; and consequently, in that Case, the Increments Be and cb will come indefinitely near to measure the Ratio of the Fluxions of the Absciss and Ordinate AC , and CB , or the Velocities with which they flow at the Point B : or because, if we suppose the Particles of Time in which any Increments are generated to be indefinitely small, the Acceleration or Retardation of the Fluxion or Velocity with which they are generated will be so too; therefore they are indefinitely

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finitely near in Proportion to the Fluxions of the Quantities of which they are Increments; and therefore (because when any Quantity is increas'd or decreas'd, but by only an infinitely or indefinitely small Particle, that Quantity may be consider'd as remaining the same as it was before;) these Increments may be taken as Proportional to, or for the Fluxions in all Operations; and, on the contrary, the Fluxion for the Increment.

4. *Note*, Those Quantities which are suppos'd to flow, or be generated by continual Increase, are call'd *Fluents*, and *variable* or *flowing* Quantities; and those which admit of no Variation, are call'd, *fixt*, *given*, and *invariable* Quantities.

—The Beginning of the Alphabet, *viz.* *a*, *b*, *c*, *d*, *e*, &c. is used to express invariable Quantities: and the End, *viz.* *z*, *y*, *x*, &c. variable or flowing Quantities: and the *Fluxions* of those variable Quantities *z*, *y*, *x*, are denoted by $\dot{z}$, $\dot{y}$, $\dot{x}$; and their *Second Fluxions*, or the *Fluxions* of $\dot{z}$, $\dot{y}$, $\dot{x}$, by $\ddot{z}$, $\ddot{y}$, $\ddot{x}$; their *Third Fluxions*, or the *Fluxions* of $\ddot{z}$, $\ddot{y}$, $\ddot{x}$, by $\dddot{z}$, $\dddot{y}$, $\dddot{x}$, and so on: Also the *Moment*, *Increment* or *Decrement* of *z* is denoted by z' , of *y* by y' , of *x* by x' ; and the *Second Moments*, *Increments* or *Decrements* of *z*, *y*, *x*, or the *Moments Increments* or *Decrements* of z' , y' , x' , by z'' , y'' , x'' and so on.

The

DOCTRINE of FLUXIONS.

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The Fluxions and Increments of a, b, c , &c. viz. of invariable Quantities are $= 0$.

5. Note. Those Fluents which are suppos'd to flow together, or in equal Times, are call'd ^{the General} Contemporay Fluents; and the Fluxions of ^{In the same} these Contemporay Fluents are call'd Contem- ^{Times} porary Fluxions: So that if two or more of these Contemporay Fluents are equal, or in any ~~certain~~ Ratio to each other, their Fluxions ^{Constant} will be likewise equal, or in the same Propor- ^{Ratio} tion.

BEFORE we proceed to the finding of the Fluxions of Fluents (which is the Business of the Direct Method of Fluxions,) it may not be improper to premise a few Things concerning the new Method of Notation in Algebra.

6. To render the Use of Surds as little trou-
blesome as possible, and to accommodate them
to fluxional Operations; the Index of the
Power to which any Quantity is to be rais'd, is
plac'd as the Numerator of a Fraction, whose
Denominator is the Root to be extracted: Thus
the square Root of the Cube of a , which ac-
cording to the old or common Way of Nota-
tion is express'd by $\sqrt{aaa}$, is here express'd $a^{\frac{3}{2}}$ or
 $a^{\frac{3}{2}}$ and the Cube Root of $ax - xx$, which us'd
to be express'd thus $\sqrt[3]{ax - xx}$ is here express'd
thus

thus $\sqrt[n]{ax - x^n}$ Also the Square Root of x , or $\sqrt{x}$ is thus express'd, $x^{\frac{1}{2}}$

The Reason of which is plain if we consider that the Index or Exponent of the Power of any Quantity, x , is always equal to the Number of the Place that Power bears in a Geometrical Progression whose first Term and common Multiplier is the Quantity itself; or, in other Words, is equal to the Number of Times which the Quantity x must be multiplied into itself in order to produce that Power: that is, if we take the Geometrical Progression of 1: x : xx : xxx &c. and make an Arithmetical Progression of 0. 1. 2. 3 &c. then the Numbers in the Arithmetical Progression will be the Indices or Exponents of the corresponding Terms in the Geometrical Progression, and therefore may be consider'd as the Logarithms of these Terms From hence it follows that the Arithmetical Mean between 1 and 9, for Instance, *i. e.* 5, will be the Index of the Geometrical Mean between x or x^1 & x^9 which is x^5 . The Arithmetical Mean between 0 and 10 will be $\frac{1}{2}$ or 5 so the Geometrical Mean between x^0 or 1 & x^{10} will be $x^{\frac{1}{2}}$ or x^5 , that is $\sqrt{x^{10}} = x^5$. Thus likewise $\sqrt{x^2}$ will be express'd by $x^{\frac{1}{2}}$ & $\sqrt[n]{a - x^n}$ by $\sqrt[n]{a - x^n}$ &c. For the same Reason also if we take a descending Series as xx : x : 1 : $\frac{1}{x}$: $\frac{1}{x^2}$ &c. & take the Arithmetical Series 2. 1. 0.

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—1.—2.—3. &c. the Number in any Place of the Arithmetical Series will be the Index of the corresponding Place in the Geometrical

Series: Thus the Index of $\frac{1}{x^3}$ will be —3, that

is $\frac{1}{x^3}$ is $= x^{-3}$. Thus, also $\frac{x^2}{x}$ is $= x^{2-1} = x^1$

$\frac{x^1}{x} = x^0 = 1$, $\frac{1}{x^m} = x^{-m}$, and so on. In the

Exponents of Powers, Addition has the Effect of Multiplication on the respective Roots; and Multiplication of Involution, and *e contra*, Substraction of Division, and Division of Evolution, or in a Word, Exponents of Powers are entirely Logarithmical with regard to

their Roots. So that $\frac{x^{\frac{2}{3}}}{x^{\frac{1}{3}}}$ is $= x^{\frac{2}{3}-\frac{1}{3}} = x^{\frac{1}{3}}$, $\overline{x^a+x}^{\frac{1}{3}}$

is $= x^{3a+3x}$, $\overline{x^a+x}^{\frac{1}{3}}$ is $= x^{\frac{a+x}{3}}$. And $\frac{x^m}{x^n}$ is $=$

x^{m-n} . And $\frac{\overline{x^m+x}^a}{x^a}$ is $= x^{am+an-\frac{m}{a}}$.

If what has been said, be duly consider'd, no Difficulty in the new Notation will occur; and the Knowledge of it is absolutely necessary in order to understand the following Chapters.

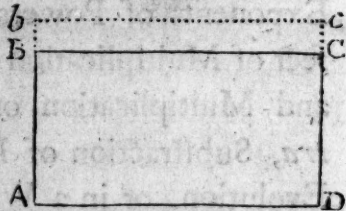
C

CHAP.

C H A P. II,

Of finding the FLUXION of a given FLUENT.

7. **R**ULE 1. To find the Fluxion of a simple Fluent wherein there is but one variable Quantity. Mark the variable Quantity, or flowing Term with a Dot over it: Thus the Fluxion of ax is $a\dot{x}$. For,

Let $AD = BC = a$, b
and $AB = DC = x$,
and let bc be suppos'd
equal and indefinitely
near and parallel to BC . A 

i. e. let $Bb = Cc = x'$. Then will $CB \times Bb = ax'$ be the Moment of the Rectangle AC or ax ; and because $\dot{x}$ may be substituted for x' (by *Art.* 3.) $a\dot{x}$ will be = its Fluxion.

8. **R**ULE 2. To find the Fluxion of a Quantity compounded of different simple Quantities or Fluents connected together with the Signs $+$ and $-$. Find the Fluxion of each simple Fluent by Rule 1st, then connect these Fluxions together with the Signs of their respective Fluents, and you will have the Fluxion of the Quantity requir'd. Thus the Fluxion of $a^2 + ax - by$ is $0 + a\dot{x} - b\dot{y}$. For since x and y flow or increase together; when ax is increas'd

DOCTRINE of FLUXIONS. 11

increas'd by its Moment ax' ; that is, when ax becomes $ax+ax'$; $-by$ will become $-by-by'$; and a^2 will remain the same, as being invariable, and consequently its Moment will be $=0$: therefore, then the Quantity $a^2+ax-by$ with its indefinitely small Increase or Decrease will become $a^2+ax-by+ax'-by'$ and the Increase or Decrease alone will be $=ax'-by'$. Wherefore, because x' and y' are expressive of $\dot{x}$ and $\dot{y}$ (*Art. 3.*) $a\dot{x}-b\dot{y}$ is $=$ the Fluxion of $a^2+ax-by$.

Again, the Fluxion of $x-y+cz$ is $=\dot{x}-\dot{y}+\dot{c}\dot{z}$. And the Fluxion of a^2x+bz is $=a^2\dot{x}+b\dot{z}$.

9. RULE 3. To find the Fluxion of the Product of two or more Quantities drawn into each other. Multiply the Fluxion of each Quantity separately by the other, or Product of the Rest; and the Sum of these Products will be the Fluxion requir'd. Thus the Fluxion of xy is $\dot{x}y+x\dot{y}$, and the Fluxion of axy is $\dot{a}xy+a\dot{x}y+ax\dot{y}$: Also the Fluxion of xyz is $\dot{x}yz+x\dot{y}z+xy\dot{z}$; and the Fluxion of $axyz$ is $\dot{a}xyz+a\dot{x}yz+ax\dot{y}z+axy\dot{z}$.

1. That the Fluxion of xy is $\dot{x}y+x\dot{y}$ may be thus demonstrated:

Let the Rectangle $AC=xy$ be a Fluent or $\dot{H}$ $\dot{K}$ flowing Quantity, *i. e.* let it increase by a continual Enlargement of its Dimention, or Sides $AB=x$ and $BC=y$. Now in order to represent this increase in its nascent State, let the prick'd Lines ad and dc be drawn very near and parallel to AD and DC ; and let CD and AD be continued on to H and K ; then Aa or DH and Cc or DK will represent the Increments of the Lines CD and AD , and $adccDA$ the Moment or Increment of the whole Rectangle AC . Now if $DK=y'$ and $DH=x'$, the whole increased Rectangle will be $=xy+xy'+x'y+x'y'$ and consequently the whole Increment or Moment is $=xy'+x'y+x'y'$: But in the very first Moment of the Existence of this Increment, or just as it is coming into Being, or beginning to be, $x'y'$ bears no assignable Ratio to either $x'y$ or xy' (for as $x'y':x'y::y':y$ and y' by Supposition is infinitely less than y , *ergo*, &c.) and therefore may very well be expunged or rejected: So that $x'y+xy'$ will express the Moment or Increment of the above Rectangle as soon as it begins to be; and then only it is that the Increment

crement is expreſſive of the Fluxion, or of the Velocity with which the Rectangle flows at the Point D: Hence by ſubſtituting the Fluxion for the Increment (*ſee Art. 3.*) we have $\dot{xy} + x\dot{y} =$ the Fluxion of the Rectangle xy .

2. Much after the ſame Manner it may be demonſtrated, that the Fluxion of axy is $\dot{axy} + ax\dot{y}$. Thus let x' expreſs the very firſt Moment of x , or the Increaſe of x juſt as it begins to be, and y' of y . Now when x becomes $x+x'$, y will become $y+y'$, but a will remain the ſame as being invariable; and therefore axy with indefinitely ſmall increaſe will become $= a(x+x')\overline{(y+y')} = axy + ax'y + axy' + ax'y'$ and its increaſe alone $= ax'y + axy' + ax'y'$: But as $ax'y :: ax'y' :: y :: y'$; and therefore, becauſe y is infinitely greater than y' , $ax'y$ is infinitely greater than $ax'y'$: So that (becauſe when any Quantity is increaſed but by an indefinitely ſmall Particle, that Quantity may be conſidered as remaining the ſame as before) the above Increment or Increaſe of the Fluent axy , may very well be conſidered as $= ax'y + axy'$, and by ſubſtituting the Fluxion for the Increment (*Art. 3.*) we have its Fluxion $= \dot{axy} + ax\dot{y}$.

3. Alſo that the Fluxion of xyz is $\dot{xyz} + x\dot{y}z + xy\dot{z}$ may be proved after the ſame Manner; thus, when x flows and becomes $x+x'$, then y and z will flow, and become equal to $y+y'$ and

$$z+z'$$

$z+z'$; and therefore then the Fluent xyz will become $=\overline{x+x'} \times \overline{y+y'} \times \overline{z+z'} = xyz + x'y'z + xy'z + xyz' + x'y'z' + xy'z' + x'y'z'$ from which subtracting the said Fluent xyz , we shall have its Moment or Increment alone $=x'y'z + xy'z + xyz' + x'y'z' + xy'z' + x'y'z'$ which may very justly be considered and taken as $=x'y'z + xy'z + xyz'$ since the Ratio of this to $x'y'z + xy'z' + x'y'z'$ is indefinitely great, or greater than any that can be assigned, as may be proved thus, $x'y'z : x'y'z' :: y : y'$, and $xy'z : xy'z' :: z : z'$, and $xyz' : x'y'z' :: x : x'$, and $x'y'z : x'y'z' :: yz : y'z'$; therefore, (because y is infinitely greater than y' , z than z' , x than x' , and yz than $y'z'$) $x'y'z$ is infinitely greater than $x'y'z'$ and $x'y'z'$, $xy'z$ than $x'y'z'$, and xyz' than $x'y'z'$; and therefore &c. wherefore, because (*Art. 3.*) x' is expressive of x , and y' of y and z' of z , by substituting the Fluxion for the Increment, we have the Fluxion of $xyz = \dot{x}yz + x\dot{y}z + xy\dot{z}$. Or thus, substitute v for xy ; then by Demonst. 1. will $\dot{v} = \dot{x}y + x\dot{y}$, and the Fluxion of $xyz = \dot{v}z + v\dot{z}$; therefore, by restitution or writing $\dot{x}y + x\dot{y}$ instead of $\dot{v}$, and xy instead of v , we shall have $\dot{x}yz + x\dot{y}z + xy\dot{z} =$ the Fluxion of xyz , as before.

4. And thus, also it may be proved, that the Fluxion of $axyz$ is $=axyz + ax\dot{y}z + axy\dot{z}$; for substituting v for axy we shall have by Dem. 2.

$$\dot{v} =$$

$\dot{v} = axy + ax\dot{y}$, and the Fluxion of $vz = \dot{v}z + v\dot{z} =$ the Fluxion of $axyz$, which by Restitution is $= axyz + ax\dot{y}z + axy\dot{z}$.

Hence,

10. The Fluxion of xx is $= \dot{x}x + x\dot{x}$, the Fluxion of xxx is $= \dot{x}xx + x\dot{x}x + xx\dot{x}$; that is, the Fluxion of x^2 is $= 2x\dot{x}$, the Fluxion of x^3 is $3x^2\dot{x}$; and if we put $m = 1, 2, 3$, or any positive Number whatsoever, the Fluxion of x^m will be $mx^{m-1}\dot{x}$.

11. RULE 4. To find the Fluxion of a Fraction. Multiply the Fluxion of the Numerator into the Denominator, from the Product of which subtract the Fluxion of the Denominator drawn into the Numerator; then this Remainder divide by the Square of the Denominator, and you will have the Fluxion sought.

Thus the Fluxion of $\frac{ax}{y}$ is $= \frac{axy - yax}{y^2}$; For

make $z = \frac{ax}{y}$; then will $yz = ax$, and therefore

the Fluxion of $yz =$ that of ax (*Art. 5.*) that is, $y\dot{z} + y\dot{z} = a\dot{x}$, wherefore by Transposition

and Division $\dot{z} = \frac{a\dot{x} - y\dot{z}}{y}$ i. e. (by restitution

or writing $\frac{ax}{y}$ for z) $\dot{z} = \frac{axy - ax\dot{y}}{y^2} =$ the Fluxion

on

on of z or $\frac{ax}{y}$. Also the Fluxion of $\frac{x^3}{y^2}$ is $=$
 $\frac{3x^2\dot{x}y^2 - 2y\dot{y}x^3}{y^4}$; the Truth of which, may be

demonstrated as before, by making $z = \frac{x^3}{y^2}$:

Again the Fluxion of $\frac{1}{x}$ is $= -\frac{\dot{x}}{x^2}$; the Fluxion
 of $\frac{1}{x^2}$ is $= -\frac{2x\dot{x}}{x^4}$; the Fluxion of $\frac{1}{x^3}$ is $=$
 $-\frac{3x^2\dot{x}}{x^6}$, &c. But by the new Notation (See

Art. 6.) $\frac{1}{x}$ is $= x^{-1}$; $\frac{1}{x^2}$ is $= x^{-2}$; $\frac{1}{x^3}$ is $=$
 x^{-3} ; and $-\frac{\dot{x}}{x^2}$ is $= -x^{-2}\dot{x}$; $-\frac{2x\dot{x}}{x^4}$ or $-\frac{2\dot{x}}{x^3}$ is
 $= -2x^{-3}\dot{x}$; $-\frac{3x^2\dot{x}}{x^6}$ or $-\frac{3\dot{x}}{x^4}$ is $= -3x^{-4}\dot{x}$,
 &c. Hence therefore,

12. The Fluxion of x^{-1} is $= -x^{-2}\dot{x}$; the
 Fluxion of x^{-2} is $= -2x^{-3}\dot{x}$; the Fluxion of
 x^{-3} is $= -3x^{-4}\dot{x}$, &c. and if m represent any
 Negative Number whatsoever, the Fluxion
 of x^m will be $= mx^{m-1}\dot{x}$.

13. RULE 5. To find the Fluxion of
 any Power of a Fluent or flowing Quantity.
 Multiply the whole Term by the Index of the
 Power, then subtract 1 from the said Index,
 and multiply this whole Term into the Fluxion

on

on of the Root of the Fluent; and you will have the Fluxion of the Fluent required. Thus the Fluxion of $x+x^2$ is $= 3 \times x+x^2$ $^{3-1}$ $\times \dot{x}+2x\dot{x}$ *i. e.* $= 3 \times x+x^2$ $^2 \times \dot{x}+2x\dot{x}$; the Fluxion of $2ax-x^2$ $^{\frac{1}{2}}$ is $= \frac{1}{2} \times 2ax-x^2$ $^{-\frac{1}{2}} \times 2a\dot{x}-2x\dot{x}$, which is $= \frac{1}{2} \times \frac{I}{\sqrt{2ax-x^2}} \times$

$2a\dot{x}-2x\dot{x}$ *i. e.* $= \frac{a\dot{x}-x\dot{x}}{\sqrt{2ax-x^2}}$; the Fluxion of

$a+ax-x^2$ $^{\frac{2}{3}}$ is $= \frac{2}{3} \times a+ax-x^2$ $^{-\frac{1}{3}} \times$

$0+a\dot{x}-2x\dot{x}$ *i. e.* $= \frac{2}{3} \times \frac{I}{\sqrt[3]{a+ax-x^2}} \times$

$a\dot{x}-2x\dot{x}$ or $\frac{2a\dot{x}-4x\dot{x}}{3 \times \sqrt[3]{a+ax-x^2}}$: And, universally,

the Fluxion of $x^{\frac{m}{n}}$ is $= \frac{m}{n} x^{\frac{m}{n}-1} \dot{x}$; where m or n , or both, may be either Affirmative or Negative whole Numbers or Fractions; so that the Index or Exponent $\frac{m}{n}$ expresses or repre-

sents any Affirmative or Negative whole Number or Fraction whatsoever. For by *Art.* 10 and 12. if m represents any Affirmative or Negative whole Number, the Fluxion of x^m will

be $mx^{m-1}\dot{x}$; therefore, if we suppose $y=x^{\frac{m}{n}}$, or which is the same, $y^n=x^m$, (n being any

D

Affirma-

Affirmative or Negative whole Number like-
 wise) by *Art. 5.* we have $ny^{n-1}\dot{y} = mx^{m-1}\dot{x}$
 which divided by ny^{n-1} gives $\dot{y} = \frac{mx^{m-1}\dot{x}}{ny^{n-1}}$ *i. e.*

by writing $x^{\frac{m}{n}}$ for its equal y , $\dot{y} = \frac{mx^{m-1}\dot{x}}{n \times x^{\frac{m}{n}-1}}$

$$= \frac{m}{n} \times \frac{x^{m-1}\dot{x}}{x^{\frac{m}{n}-1}} = \frac{m}{n} x^{\frac{m}{n}-1} \dot{x} \text{ which is } = \text{ the}$$

Fluxion of $x^{\frac{m}{n}}$ above; therefore, &c. And
 that either m or n , or both, may represent any
 Fractions as well as whole Numbers is plain,
 since this $\frac{m}{n}$ expresses the Quotient of any whole
 Number divided by another, and may be taken
 for a new m or n ; and so on, *ad infinitum*.

14. RULE 6. To find the Fluxion of a
 Logarithm. The Fluxion of the Hyperbolic
 Logarithm of any Quantity, is equal to the
 Fluxion of that Quantity divided by the Quan-
 tity itself: Thus the Fluxion of the hyperbolic
 Logarithm of x is $= \frac{\dot{x}}{x}$; the Truth of which
 we shall hereafter Demonstrate (*See Chap. 14.*
Quest.)

Quest. 10.) then, as the hyp. Log. of 10 (*viz.* 2.3025850929 &c.) : is to the common Log. of 10 (*viz.* 1) :: so is the hyp. Log. of any Number x : to the common Log. of that same Number; that is, if we put $L=2.30258$ &c. As $L : 1 :: \text{hyp. Log. of } x : \text{common Log. of } x$, and therefore, (*Art. 5.*) :: Fluxion of hyp. Log. of x : Fluxion of common Log. of

x . Hence we have $L : 1 :: \frac{\dot{x}}{x} : \frac{\dot{x}}{Lx} =$ the Fluxion of the common Log. of x ; or, because $\frac{1}{L} = 0.4342944819$ &c. if we put this

$\frac{1}{L} = M$, then the Fluxion of the common

Log. of x , *viz.* $\frac{\dot{x}}{Lx}$, will be $= \frac{\dot{x}}{x} \times M$, *i.e.* the

Fluxion of the hyperbolic Logarithm of any Number multiplied by (M or) 0.434294 &c. gives the Fluxion of the common Logarithm of that same Number,

15. *Note*, It is sometimes expedient, in order to find the Fluxion, to divide the Equation given, by one or more of the variable Quantities contained in most of the Terms : And to substitute single Letters for compound Quantities, Thus if the given Equation were

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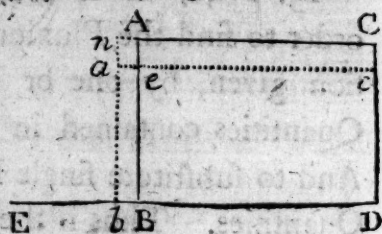
$xy +$

$xy + a - axy + y^2 = yz$, divide both Sides of the Equation by y ; then $x + \frac{a}{y} - ax + y = z$; and

the Fluxion of this Equation is $\dot{x} - \frac{a\dot{y}}{y^2} - a\dot{x} + \dot{y} = \dot{z}$.

And if the Equation given were $x - by + \frac{x^2y}{z} = az$, substitute v for $\frac{x^2y}{z}$; then $x - by + v = az$, and this in Fluxions is $\dot{x} - b\dot{y} + \dot{v} = a\dot{z}$.

16. *Note*, Hitherto we have supposed, when one variable Quantity in any Fluent increases, that the others, if any, increase likewise; but it sometimes happens, that some of them decrease while the others increase: in which Case, the Fluxions of these decreasing are Negative, with respect to those of the increasing Quantities; and therefore the Signs of the Terms which are affected with them, ought to be Negative. Thus, if whilst x increases, y decreases, the Fluxion of the Fluent xy will be $=\dot{x}y - x\dot{y}$. To prove which, let the Line AB be supposed perpendicular to the Line DE and equal and parallel to the Line CD and to move along



on

on the said Line DE towards E; and at the same Time, let the Point A be supposed to move along the Line AB towards B, so as that when the Point B is advanced to b , the Point A may be arrived at a (ab and ac being supposed indefinitely near and parallel to AB and AC;) then the little Parallelogram ACce subtracted from the other Beab will be the Moment of the Rectangle ABCD. Now if we put AC or BD= x , AB or CD= y , Increment Bb or ea= x' , and decrement Ae or na= y' ; then will eB= $y-y'$, and the little Rectangle Ba= $\overline{y-y'} \times x' = yx' - x'y'$, i. e. = the little Rectangle Bn, less the little Rectangle en; but since these two little Rectangles are to each other as BA to eA, and the Point e is supposed to be indefinitely near to the Point A, or Ae to be but just beginning to be, or coming into Being; therefore the little Rectangle Bn cannot properly be said to be diminished or lessened by the subtracting from it the little Rectangle en which is so infinitely less: Wherefore the said little Rectangle Ba may be considered as equal to yx' ; from which if we subtract the little Rectangle Ac, which is = xy' , we shall have $yx' - xy' =$ the Moment of the Rectangle xy or BC; therefore by substituting $\dot{x}$ and $\dot{y}$ for x' and y' (Art. 3.) we shall have

have the Velocity with which the said Rectangle increases or decreases at the Point A, or the Fluxion of it at that Point $= yx - xy$. Again, if z decreases whilst y and x increases, then the Fluxion of the Fluent xyz will be $= xyz + xyz - xyz$. Understand the same of the Fluxion of any other Fluent of a like Nature whatsoever.

If what has been already delivered, be thoroughly understood, we hope the Learner will meet with but few Difficulties in the Application thereof; to which we now proceed.

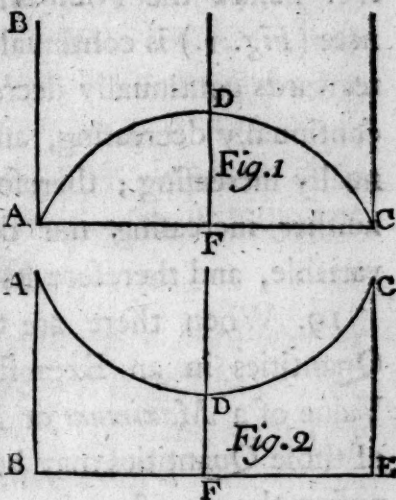
C H A P. III.

Of finding the Maxima and Minima (or greatest and least) of variable Quantities.

17. **I**F a variable Quantity is of such a Nature that it increases continually without end, or decreases till it vanishes, its greatest or least Magnitude cannot be assigned or determined; but if there is a certain Limit beyond which it cannot pass, and it is determinable when it arrives at that Limit; or, if at first it increases till a certain assignable Term and then decreases, or first decreases, and then increases; its Magnitude at such Term, is called a *Maximum*

thum or *Minimum*: Now every such *Maximum* or *Minimum*, must be in its own Nature a stable or invariable Quantity; and therefore, because an invariable Quantity has no Fluxion, its Fluxion is equal to nothing.

18. To illustrate and make this plain-
er: Let the right Line BA be supposed perpendicular to the right Line AC (*Fig. 1.*) or BE (*Fig. 2.*) and to be carried along the said Line with any given Velocity, and in such a Manner,



as that it always be parallel to its first Situation; and, at the same time, let a Point be conceived to move along upon the Line AB , from A , with such Velocity, as that it be always in the Curve ADC : Then will the Velocity with which the Line BA moves along the Line AC , or BE , be the Fluxion of the Abscissa; the Velocity with which the Point moves along the Line AB , be the Fluxion of the Ordinate; and the Velocity with which the Point moves along the Curve, be the Fluxion

Fluxion of the Curve. Now when the Point, moving from A towards B, arrives at D, it can proceed no farther, but must return towards A again; because the Ordinate FD (*Fig. 1.*) is the greatest, or (*Fig. 2.*) the least possible, *i. e.* before the Point arrives at D, the Ordinate (*Fig. 1.*) is continually increasing, and afterwards continually decreasing, or (*Fig. 2.*) is continually decreasing, and afterwards, continually increasing; therefore at that Point, it is neither increasing nor decreasing, but is invariable, and therefore its Fluxion is $=0$.

19. When there are two or more variable Quantities in an Expression, representing the Value of a *Maximum* or *Minimum*, the Value of those Quantities may be determined by supposing them to flow separately, or one by one, whilst the rest are considered as invariable. For Instance, if it were required to find the greatest Parallelopipedon, that can be contained under a given Superficies, it is evident that each of its Sides must separately flow out to a certain determinate Point, beyond which it must not pass; and therefore its Fluxion at that Point must cease, or become $=0$; or, because if the Fluxion of the given Expression of the *Maximum* or *Minimum*, when only one of the Quantities is considered as variable, be

not

be not $=0$, the same Expression may become greater or less without varying the Values of those which are considered as constant; therefore, when it is the greatest or least possible, each of those Fluxions must then become $=0$.

20. *Note, 1.* In curvilinear Spaces, it is in effect the same Thing to seek the greatest Area that can be contained under a given Perimeter, as to seek a given Area under the least Perimeter: And the same holds good, in respect of Solids and their Surfaces.

Note, 2. If any Quantity is a *Maximum* or *Minimum*, all the Powers of it will be so too; as will also the Product or Quotient, arising by its being multiplied or divided by any invariable or given Quantity. Thus, if $\frac{ab}{c} \times \sqrt{x^2 - 2ax}$ represent a *Maximum* or *Minimum*, then $x^2 - 2ax$ will be also a *Maximum* or *Minimum*, and consequently its Fluxion must be $=0$.

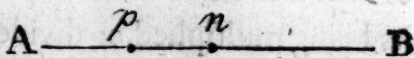
Note 3. Generally, when a variable Quantity admits of a *Maximum*, its *Minimum* is nothing; and when it admits of a *Minimum*, its *Maximum* is infinite.

EXAMPLE I.

21. To find the Value of x , in Terms of a and b , when $bx - ax^2$ is a Maximum.

BECAUSE the Fluxion of a *Maximum* is $= 0$, therefore the Fluxion of $bx - ax^2$ must be $= 0$; that is $b\dot{x} - 2ax\dot{x} = 0$. Now if we divide this Equation by $\dot{x}$, we shall have $b - 2ax = 0$; therefore, by Transposition, we have $2ax = b$ and by Division $x = \frac{b}{2a}$ which is the Value of x required.

EXAMPLE II.

22. To find the  Point p in the given right Line AB , where the Rectangle Ap into pB is a Maximum, or greater than any other Rectangle An into nB .

PUT the given right Line $AB = a$, and $Ap = x$; then $pB = a - x$, and by the Data $Ap \times pB = x \times a - x = ax - x^2 = a \text{ Maximum}$; therefore the Fluxion of $ax - x^2$ is $= 0$, that is, $a\dot{x} - 2x\dot{x} = 0$, which divided by $\dot{x}$ gives $a - 2x = 0$, therefore $2x = a$, and $x = \frac{1}{2}a$. Consequently

quently the Rectangle of the Parts $A p$ and $p B$ are the greatest when those Parts are equal.

Or thus, Put $A p = x$, and $p B = y$; then $A p \times p B = xy = a \text{ Maximum}$: Now it being evident that if x increase, y must decrease, therefore (*Art. 16.*) $\dot{x}$ and $\dot{y}$ are Negative to each other; and consequently the Fluxion of xy , when a *Maximum*, is $\dot{x}y - x\dot{y} = 0$: But it is likewise evident, that the Increment of x is equal to the Decrement of y , or $\dot{x} = -\dot{y}$; therefore strike both out in the above Fluxion of xy ; then we shall have $y - x = 0$, and therefore $x = y$, as before.

EXAMPLE III.

23. To find the Point $A \text{---} p \text{---} B$
p in the given right

Line AB , when $\overline{A p}^m \times \overline{p B}^n$ is a Maximum.

PUT the given right Line $AB = a$, and $A p = x$; then $p B = a - x$, and $\overline{A p}^m \times \overline{p B}^n = x^m \times a - x^n = a \text{ Maximum}$, therefore its Fluxion is $= 0$; that is, (*see Art. 13. and 9.*) $m x^{m-1} \dot{x} \times a - x^n + n \cdot a - x^{n-1} \times -\dot{x} \times x^m = 0$ (*i. e.*) $m x^{m-1} \dot{x} \times a - x^n - n a - x^{n-1} \dot{x} \times x^m = 0$, which divided by $x^{m-1} \dot{x}$ gives $m a - x^n - n a - x^{n-1} x = 0$; therefore $m a - x^n = n a - x^{n-1} x$; then

E 2

divide

divide by $\overline{a-x}^{n-1}$ and it will be $m \cdot \overline{a-x} = nx$, i. e. $ma - mx = nx$, or $mx + nx = ma$, therefore $x = \frac{m}{m+n} a$, and the Point p is determined.

Or thus, Put $Ap = x$, and $pB = y$; then $y^n x^m = a$ Maximum, which in Fluxions is $ny^{n-1} y x^m - mx^{m-1} x y^n = 0$, (see Art. 16.) but $\dot{x}$ and $\dot{y}$ are equal (Art. 22.) therefore throw both $\dot{x}$ and $\dot{y}$ = then $ny^{n-1} x^m - mx^{m-1} y^n = 0$. And $ny^{n-1} x^m = mx^{m-1} y^n$. Now by dividing both Sides of this Equation by $y^{n-1} x^{m-1}$ we shall have $nx = my$ therefore $m : n :: x : y$. So that the Segments are in direct Proportion to the Powers in the Maximum.

EXAMPLE IV.

24. To find the Values of x , y , and z , when $x + y + z = a$ and $xy + z^2 = a$ Minimum.

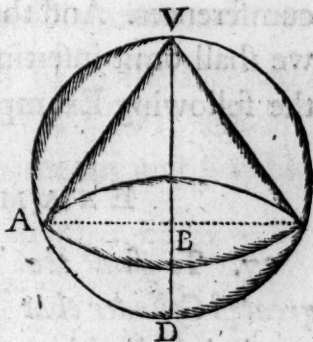
BECAUSE $x + y + z = a$, therefore $x = a - y - z$, which substituted for x , gives $xy + z^2 = ay - y^2 - yz + z^2 = a$ Minimum. Now by Art. 19. we may consider y as variable whilst z remains invariable, and z as variable whilst y remains invariable; therefore, by supposing z invariable, the Fluxion of the above Expression of the Minimum will be $ay - 2yy - yz = 0$,
and

and by supposing y invariable, it will be $-y\dot{z} + 2z\dot{z} = 0$: Now by dividing the First of these two fluxional Equations by $\dot{y}$ we have $a - 2y - z = 0 \therefore z = a - 2y$, and by dividing the Second of them by $\dot{z}$ we have $-y + 2z = 0 \therefore z = \frac{1}{2}y$. Hence $\frac{1}{2}y = a - 2y$, and therefore $y = \frac{2}{5}a$, and consequently $z = (\frac{1}{2}y) = \frac{1}{5}a$, and $x = (a - y - z) = \frac{2}{5}a$.

EXAMPLE V.

25. To find a Cone inscribed in a given Sphere, whose convex Surface shall be a Maximum.

PUT the Diameter of the given Sphere $(VD) = a$, the Altitude of the inscribed Cone $(VB) = x$, and $3.14159 \&c. = c$; then $BD = a - x$. Now, by a well known



$\times BD = \overline{BA}^2$; that is, $x \times a - x = ax - x^2 = \overline{BA}^2$: And by 47 E. 1. $AV = \sqrt{\overline{VB}^2 + \overline{BA}^2}^{\frac{1}{2}}$, i. e. $AV = \sqrt{x^2 + ax - x^2}^{\frac{1}{2}} = \sqrt{ax}^{\frac{1}{2}}$. But $BA \times c = \sqrt{ax - x^2}^{\frac{1}{2}} \times c = \sqrt{c^2 ax - c^2 x^2}^{\frac{1}{2}} = \frac{1}{2}$ the Circumference of the Cone's Base, which drawn into its slant Height $(AV =) \sqrt{ax}^{\frac{1}{2}}$, gives $\sqrt{c^2 ax - c^2 x^2}^{\frac{1}{2}} \times \sqrt{ax}^{\frac{1}{2}}$

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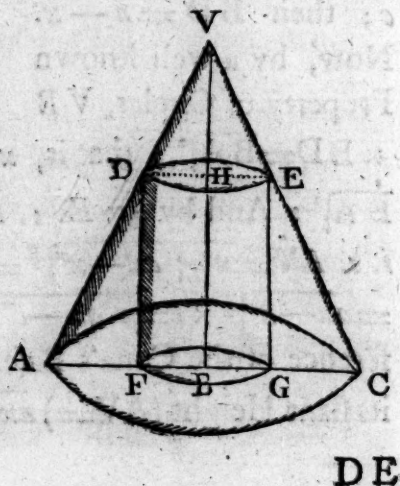
$\times \overline{ax}^{\frac{1}{2}} = \overline{c^2 a^2 x^2 - c^2 ax^3}^{\frac{1}{2}} =$ its Curve Superficies, which by Quest. must be a *Maximum*; therefore $c^2 a^2 x^2 - c^2 ax^3 =$ the Square of a *Maximum*, or a *Maximum* also (*Art. 20, Note 2.*) and its Fluxion, therefore $2c^2 a^2 xx - 3c^2 ax^2 x = 0$, which divided by $c^2 axx$ gives $2a - 3x = 0$; therefore $3x = 2a$ and $x = \frac{2}{3}a$. So that the Convex Surface of the Cone will be the greatest, when its Altitude is $= \frac{2}{3}$ of the Sphere's Diameter.

26. *N. B.* There was no need of introducing c in the above, for $VA \times AB =$ a *Maximum*; the Radii of Circles being as their Circumferences. And therefore in such like Cases, we shall omit inserting it as often as may be in the following Examples.

EXAMPLE VI.

27. To find the greatest Cylinder that can be inscribed in a given right Cone e.g. whose Altitude BV is $= a$, and Base-diameter $AC = b$.

PUT the Diameter of the Cylinder



DOCTRINE of FLUXIONS. 31

DE or FG = x ; then, the Δ 's VAC and VDE being Similar, as AC : BV :: DE : HV, i. e. $b : a :: x : \frac{ax}{b} = HV \therefore HB = BV$

—HV = $a - \frac{ax}{b}$ = the Altitude of the Cylinder. Whence (see Art. 26.) $\overline{FG}^2 \times BH =$

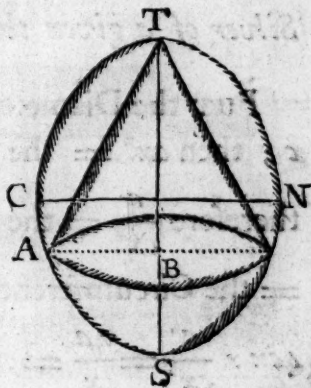
$$x^2 \times a - \frac{ax}{b} = \frac{abx^2 - ax^3}{b} = \text{a Maximum. Con-}$$

sequently (see Art. 20. Note 2.) $bx^2 - x^3 = \text{a Maximum}$, in Fluxions $2bx\dot{x} - 3x^2\dot{x} = 0$, which divided by $x\dot{x}$ gives $2b - 3x = 0 \therefore 3x = 2b$ and $x = \frac{2}{3}b = FG$ or DE. Consequently BH (= a

— $\frac{ax}{b}$) = $\frac{1}{3}a$. So that the Diameter and Altitude of the Cylinder, will be to the Diameter and Altitude of the Cone, as $\frac{2}{3}$ to 1, and as $\frac{1}{3}$ to 1, when the Cylinder is the greatest possible.

EXAMPLE VII.

28. To find the largest Cone that can be inscribed in a given prolate Spheroid, viz. whose Transverse and Conjugate Diameters TS and CN are a and b .



Put the Altitude of

the

the inscribed Cone $BT=x$; then $BS=TS=TB=a-x$. Now by a well known Property of Ellipses $TS^2 : CN^2 :: TB \times BS : AB^2$

$$i. e. a^2 : b^2 :: x \times a - x : \frac{ab^2x - b^2x^2}{a^2} = BA^2$$

which (*Art.* 26.) drawn into $(\frac{1}{3}BT) \frac{1}{3}x$ gives

$$\frac{ab^2x^2 - b^2x^3}{3a^2} = a \text{ Maximum.} \text{ Consequently,}$$

(*Art.* 20. *Note* 2.) $ax^2 - x^3 = a \text{ Maximum,}$ which in Fluxions is $2axx - 3x^2x = 0$; then divide by xx and it gives $2a - 3x = 0$. Consequently $3x = 2a$ and $x = \frac{2}{3}a$, that is, the Altitude of the Cone must be $= \frac{2}{3}$ of the Diameter of the Spheroid.

EXAMPLE VIII.

29. To find the internal Dimensions of a cylindrical Cup, whose Capacity is given $=a$; when made with the least possible Quantity of Silver of a given thickness.

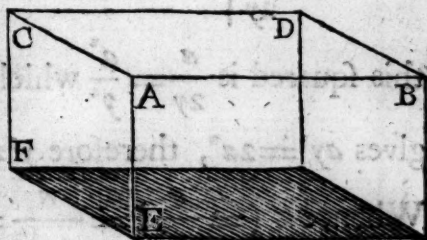
Put the Diameter $=x$, and $.78539 \&c. = c$; then $cx^2 =$ the Area of the Bottom, and therefore $\frac{a}{cx^2} =$ the Altitude: But $4cx$ being $=$ the Circumference of the Bottom, therefore $4cx \times \frac{a}{cx^2} = \frac{4a}{x} =$ the inside Curve Surfaces.

ficies[†]; which, because the Quantity of Silver ^{† Hence} is the least possible, is a *Minimum*. And this $cx^2 + \frac{4a}{x} =$ ^{Whole Area} therefore in Fluxions is $2cx\dot{x} - \frac{4a\dot{x}}{x^2} = 0$, which ^{Superficie} multiplied by x^2 gives $2cx^3\dot{x} - 4a\dot{x} = 0$, and this divided by $2\dot{x}$ gives $cx^3 - 2a = 0$; therefore $cx^3 = 2a$, and $x = \sqrt[3]{\frac{2a}{c}}$.

EXAMPLE IX.

30. To find the internal Dimensions of a Cistern, in the Form of a rectanglar Solid, i. e. whose Bottom and all four Sides are rectanglar; when its Capacity is $= a$ and made with the least possible Quantity of Lead of a given thickness.

Put the inside Length AB or $CD = x$, Breadth AC or $BD = y$, and Depth AE or $CF = z$; then



$xyz = a$, and therefore $x = \frac{a}{yz}$. Now the in-

side Superficies of the Bottom and four Sides is $= CD \times DB + 2BA \times AE + 2CA \times AE$, i. e.

F

$= xy$

$\text{---}xy + 2xz + 2yz$ (or, by substituting $\frac{a}{yz}$ for x ,) $= \frac{a}{z} + \frac{2a}{y} + 2yz$, which, because the Cistern is made of the least possible Quantity of Lead, is equal a *Minimum*; and this therefore, by making y and z flow separately (*see Art. 19.*) in Fluxions is $2yz - \frac{2ay}{y^2} = 0$, and $2yz - \frac{az}{z^2} = 0$; the first of which Equations multiplied by y^2 gives $2yzy^2 - 2ay = 0$, and this divided by $2y$ gives $zy^2 - a = 0$. $\therefore z = \frac{a}{y^2}$. And the second Equation multiplied by z^2 gives $2yz^2 - az = 0$, and this divided by z is $2yz - a = 0$. $\therefore z = \frac{a}{2y}$. Hence we have $\frac{a}{2y} = \frac{a}{y^2}$ and this squared is $\frac{a}{2y} = \frac{a^2}{y^4}$ which multiplied by $2y^4$ gives $ay^3 = 2a^2$, therefore $y^3 = 2a$ and $y = \sqrt[3]{2a}$. Whence $z (= \frac{a}{y^2}) = \frac{a}{2a^{\frac{2}{3}}} = \frac{2a}{2 \times 2a^{\frac{2}{3}}} = \frac{1}{2} \times 2a^{\frac{1}{3}}$. And $x (= \frac{a}{yz}) = \frac{a}{2a^{\frac{1}{3}} \times \frac{1}{2} 2a^{\frac{1}{3}}} = \frac{a}{a^{\frac{2}{3}}} = a^{\frac{1}{3}} \times 2a^{\frac{1}{3}} = 2a^{\frac{1}{3}}$. So that $x = y = 2z$, that is, the
Length

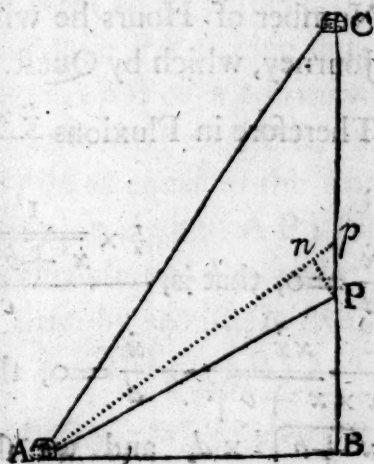
Length and Breadth will be equal, and each equal to twice the Depth.

Or thus, Let the inside Length = x , Breadth = y , and Depth = z ; then the Superficies of the Bottom and four Sides = $xy + 2xz + 2yz =$ a Minimum, as before; or, by substituting b for x , it will be $= by - 2bz + 2yz$. In Fluxions $-by - 2bz + 2yz + 2yz = 0$. And taking the homologous Terms * $2yz - by = 0$, and $2yz - 2bz = 0$; by Division and Transposition $2z = b$, and $y = b$, therefore, by Restitution $x = y = 2z$. But $xyz = a$, i. e. $\frac{1}{2}x^3$, or $\frac{1}{2}y^3 = a \therefore x$ or $y = 2a^{\frac{1}{3}}$ and $z = \frac{1}{2}2a^{\frac{1}{3}} = \frac{1}{2}a^{\frac{1}{3}}$.

* This in Effect is the same as making y and z flow separately according to Art. 19.

EXAMPLE X.

31. A Gentleman wants to ride from the City A to the City C, the Cities being a Miles apart; now from A to B, which is b Miles, or from A to any Place P in the Road CB which is perpendicular to the Road BA



he can ride after the Rate of c Miles an hour, but from B to C he can ride faster, or after the rate of d Miles an hour : To find P, the Place to which he must directly ride in order to perform his Journey in the least possible Time.

By 47 E. 1. $BC = \sqrt{CA^2 - AB^2}^{\frac{1}{2}} = \sqrt{a^2 - b^2}^{\frac{1}{2}}$ which put $= e$; also let $BP = x$, then $PC =$

$e - x$; therefore by Quest. $\frac{e-x}{d} =$ the Number of Hours he will be riding from P to C.

Again by 47 E. 1. $AP = \sqrt{PB^2 + BA^2}^{\frac{1}{2}} = \sqrt{x^2 + b^2}^{\frac{1}{2}}$, therefore by Quest. $\frac{\sqrt{x^2 + b^2}^{\frac{1}{2}}}{c} =$ the

Number of Hours he will be riding from A to

P. Hence we have $\frac{\sqrt{x^2 + b^2}^{\frac{1}{2}}}{c} + \frac{e-x}{d} =$ the

Number of Hours he will be performing his Journey, which by Quest. must be a *Minimum*:

Therefore in Fluxions $\frac{\frac{1}{2} \times \sqrt{x^2 + b^2}^{-\frac{1}{2}} \times 2x\dot{x}}{c} -$

$\frac{\dot{x}}{d} = 0$, that is, $\frac{\frac{1}{2} \times \frac{1}{\sqrt{x^2 + b^2}^{\frac{1}{2}}} \times 2x\dot{x}}{c} - \frac{\dot{x}}{d} = 0$, or

$\frac{x\dot{x}}{c \times \sqrt{x^2 + b^2}^{\frac{1}{2}}} - \frac{\dot{x}}{d} = 0$, then multiply by $c \times \sqrt{x^2 + b^2}^{\frac{1}{2}} \times d$, and we shall have $dx\dot{x} - c \times \sqrt{x^2 + b^2}$

$\sqrt{x^2 + b^2}^{\frac{1}{2}} \dot{x} = 0$, and this transposed and divided by $\dot{x}$ gives $dx = c \times \sqrt{x^2 + b^2}^{\frac{1}{2}}$ which squared is $d^2 x^2 = c^2 x^2 + c^2 b^2$, therefore $d^2 x^2 - c^2 x^2 = c^2 b^2$ and $x^2 = \frac{c^2 b^2}{d^2 - c^2}$ consequently $x = \frac{cb}{\sqrt{d^2 - c^2}^{\frac{1}{2}}} =$ BP.

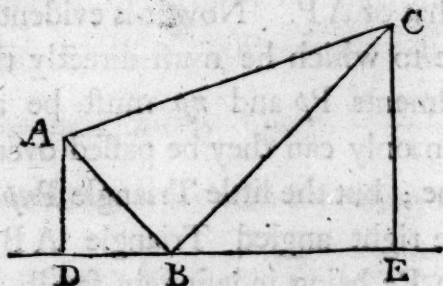
Or thus, Let Ap be supposed indefinitely near to AP , and the little circular Arch Pn described with the Radius AP ; that is, let Pp express the indefinitely small Increment of BP and np that of AP . Now it is evident, if P be the Place to which he must directly ride, that the Increments Pp and np must be as d to c , since then only can they be passed over in the same time; but the little Triangle Pnp is Similar to the right angled Triangle ABP ; (for the Arch Pn being indefinitely small, may be considered as a little right Line perpendicular to Ap ; and so the Angle PAp being indefinitely little, the Angle ApB or npP may be considered as equal to the Angle APB , and therefore the Angle pPn as equal to the Angle PAB .) Consequently as $d : c :: AP : PB$. But when the Hypothenuse and Perpendicular are d and c , the Base by 47 E. 1. will be $\sqrt{d^2 - c^2}^{\frac{1}{2}}$, therefore, $\sqrt{d^2 - c^2}^{\frac{1}{2}} : (AB) b :: c : \frac{bc}{\sqrt{d^2 - c^2}^{\frac{1}{2}}} = BP$, as before.

E x -

EXAMPLE XI.

32. Let the Triangle ABC have one Angle B in the right Line DE. To find a Maximum of the Sum of its Perpendiculars AD and CE dropt from the other two Angles on the right Line aforesaid.

N. B. $AB=3=a$, $BC=4=b$, and the $\angle ABC=90^\circ$.



Put $DB=x$; then by 47 E. 1. $AD = \sqrt{AB^2 - BD^2} = \sqrt{a^2 - x^2}$. Now, because the $\angle ABC$ is right, the $\angle CBE$ is the Complement of the $\angle ABD$, and therefore is $= \angle BAD$; consequently the Triangles ABD and CBE are Similar; therefore $AB : BD :: BC : CE$, i. e. $a : x :: b : \frac{bx}{a} = CE$. Hence

we have $AD + CE = \sqrt{a^2 - x^2} + \frac{bx}{a} = a$ Maxi-

mum

num by Quest. in Fluxions $\frac{1}{2} \times \overline{a^2 - x^2}^{-\frac{1}{2}} \times$
 $-2x\dot{x} + \frac{b\dot{x}}{a} = 0$, i. e. $-\frac{x\dot{x}}{\overline{a^2 - x^2}^{\frac{1}{2}}} + \frac{b\dot{x}}{a} = 0$,

which multiplied by $\overline{a^2 - x^2}^{\frac{1}{2}} \times a$ gives $-ax\dot{x}$
 $+ \overline{a^2 - x^2}^{\frac{1}{2}} b\dot{x} = 0$: Therefore by Transpositi-
 on and dividing by $\dot{x}$ we have $ax = \overline{a^2 - x^2}^{\frac{1}{2}} b$
 by Involution $a^2 x^2 = a^2 b^2 - b^2 x^2$. Consequent-
 ly $a^2 x^2 + b^2 x^2 = a^2 b^2$, and $x^2 = \frac{a^2 b^2}{a^2 + b^2}$ there-

fore $x = \frac{ab}{\overline{a^2 + b^2}^{\frac{1}{2}}} = 2.4$. Hence AD + CE

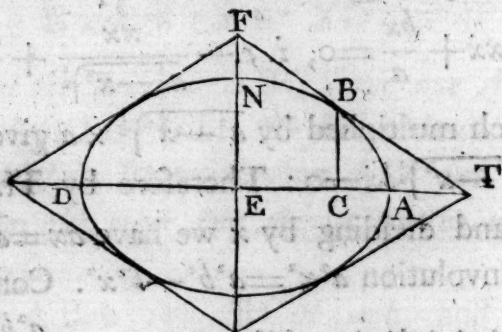
$(= \overline{a^2 - x^2}^{\frac{1}{2}} + \frac{bx}{a}) = 1.8 + 3.2 = 5 =$ the Sum

of the Perpendiculars, when that Sum is the
 greatest possible.

EXAMPLE XII.

33. To find the greatest Ellipsis that can be
 inscribed in a Rhombus, whose Diagonals are
 given.

Let the Rhombus be drawn, and the El-
 lipsis inscribed as in the Fig. and from the
 Point of Contact B the Ordinate BC let fall.
 Put ET = a , EF = b , and EA or ED = x .
 Then (see *De L'Hospital's Conic Sections Art.*



57.) as $ET : EA :: EA : EC$, *i. e.* $a : x$

$$:: x : \frac{x^2}{a} = EC \therefore CA = x - \frac{x^2}{a} = \frac{ax - x^2}{a},$$

$$CD = \frac{ax + x^2}{a}, \text{ and } CT = \frac{a^2 - x^2}{a}. \text{ By Sim.}$$

$\Delta s TE : EF :: TC : CB$, *i. e.* $a : b ::$

$$\frac{a^2 - x^2}{a} : \frac{a^2 b - bx^2}{a^2} = CB. \text{ Now by a well}$$

known Property of Ellipses $DC \times CA : \overline{CB}^2$

$$:: DE \times EA : EN^2, \text{ i. e. } \left(\frac{ax + x^2}{a} \times \frac{ax - x^2}{a} \right)$$

$$=) \frac{a^2 x^2 - x^4}{a^2} : \left(\frac{a^2 b - bx^2}{a^2} \right)^2 =)$$

$$\frac{a^4 b^2 - 2a^2 b^2 x^2 + b^2 x^4}{a^4} :: x^2 ;$$

$$\frac{a^6 b^2 x^2 - 2a^4 b^2 x^4 + a^2 b^2 x^6}{a^6 x^2 - a^4 x^4} = b^2 - \frac{b^2 x^2}{a^2} = EN^2.$$

Now, because the Ellipsis is to be the greatest possible

possible (Art. 26.) $\overline{EN}^2 \times \overline{EA}^2 = b^2 - \frac{b^2 x^2}{a^2}$

$\times x^2 = b^2 x^2 - \frac{b^2 x^4}{a^2}$ = the Square of a *Maximum*, which therefore divided by b^2 (see Art.

20. Note 2!) gives $x^2 - \frac{x^4}{a^2} =$ a *Maximum*,

which in Fluxions is $2xx - \frac{4x^3 \dot{x}}{a^2} = 0$, and this multiplied by a^2 and divided by $2xx$ gives $a^2 - 2x^2 = 0 \therefore 2x^2 = a^2$ and $x = a\sqrt{\frac{1}{2}}$.

COROLLARY.

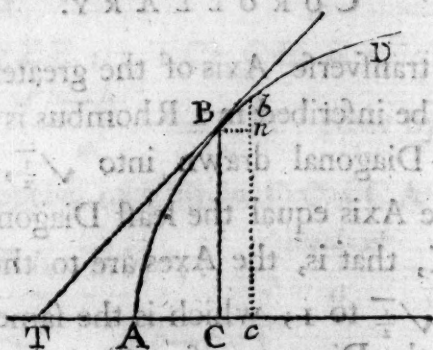
The transverse Axis of the greatest Ellipsis that can be inscribed in a Rhombus is equal the greatest Diagonal drawn into $\sqrt{\frac{1}{2}}$, and the conjugate Axis equal the least Diagonal drawn into $\sqrt{\frac{1}{2}}$, that is, the Axes are to the Diagonals, as $\sqrt{\frac{1}{2}}$ to 1; which is the same Proportion, as the Diameter of a Circle bears to the Diagonal of its circumscribing Square.

P. A. H. C.

CHAP. IV.

Of drawing TANGENTS to CURVES.

34. **A**S it is the Nature of all Curves, to have their Directions alter'd in every Point, so the Directions of every Curve will be various as the indefinite Number of Particles or Increments, of which it is composed; and to find the Direction of a Curve in any given Point, or a general Expression for its Direction in all Points, is what is meant by drawing a Tangent to it.



If the right Line **T B** be drawn to coincide with any Point **B** of the geometrical Curve **A B D** so as to touch, but not to cut it, that Line is a Tangent to the Curve in that Point: And because no two Lines can coincide, unless they have the same Direction, therefore the
Direction

Direction of the Tangent, is properly, the Direction of the Curve in the Point of Contact: And what is required, by drawing a Tangent, is generally to find the Sub-tangent CT , or the Distance from the Point C , of the Ordinate BC to the Point T , where the Axis produced, if occasion, is intersected by the Tangent.

35. Let cb be supposed parallel to CB , and Bn equal and parallel to Cc ; then if the Line cb be removed towards CB , in a parallel Motion, till it coincides with it, the Moment before its coincidence, the Triangle Bnb will be in its evanescent State; or, which is the same thing, if the said two Lines be separated from their Coincidence, the very first Moment of their Separation, produces the said Triangle in its nascent State; and in that Moment, the Line nb , terminated by the Line Bn at one End, and by the Curve at the other, comes indefinitely near to the Tangent TB produced; Consequently the Triangles bnB and BCT come indefinitely near to Similarity, and may be considered as Similar: Wherefore, then, $bn : nB :: BC : CT$, or, (putting $AC = x$, and $CB = y$) $y' : x' :: y : \frac{x'y}{y'} = CT$, that $y : x ::$

$y : \frac{\dot{x}y}{\dot{y}} = \text{CT}$. And this is a general Expression for the Subtangent of every Curve whose Abscissa is x and Ordinate y .—Now by means of the Fluxion of the Equation of the Curve, we may get the Value of $\dot{x}$ express'd in Terms that will be all affected with $\dot{y}$, which therefore being multiplied by $\frac{y}{\dot{y}}$ will give the Subtangent CT in definitive or known Terms freed from Fluxions; by which the sought Tangent BT to a given Point may be drawn.

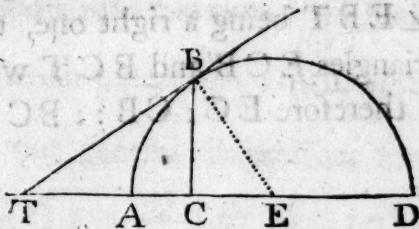
36. *Note*, When $\dot{x}$ and $\dot{y}$ are Negative to each other, (*Art.* 16.) the general Expression ($\frac{\dot{x}y}{\dot{y}}$ for the Subtangent, will be $-\frac{\dot{x}y}{\dot{y}}$, which shews that it lies on the other side of the Ordinate, with regard to the Abscissa x : For tho' $-\frac{\dot{x}y}{\dot{y}}$ may be, and really is a negative algebraic Quantity, yet it may also represent a geometrical Quantity, which is always affirmative; and as every Subtangent is in its own Nature positive, therefore the negative Sign (either in the general or definitive Expression) only shews where we must

must look for the Subtangent; that is, whether x the Absciss, or some Part of it be (as in the Affirmative it always is,) or be not (as in the Negative it never is,) included in the Expression for the Subtangent.

There are other Methods of drawing Tangents, but this is the most general and easy.

EXAMPLE I.

37. To draw a Tangent to a Circle.



Put the Absciss $AC = x$, Ordinate $CB = y$, and Radius EA or $ED = r$; then $CD = 2r - x$. Now by the Property of Circles $AC \times CD = CB^2$, i. e. $2rx - x^2 = y^2$, and this Equation put into Fluxions is $2rx - 2xx = 2yy$, which divided by $2r - 2x$ gives $\dot{x} = \frac{2y\dot{y}}{2r - 2x}$

$\frac{y\dot{y}}{r - x}$, and this multiplied by $\frac{y}{\dot{y}}$ (*Art.* 35.) or substituted for $\dot{x}$ in $(\frac{xy}{\dot{y}})$ the general Expression

fion

tion for the Subtangent, gives the Subtangent

$$CT (= \frac{xy}{y}) = \frac{y^2}{r-x} = (\text{by putting } 2rx - x^2$$

$$\text{for its equal } y^2 \text{ as above}) \frac{2rx - x^2}{r-x} = \frac{CB^2}{EC}.$$

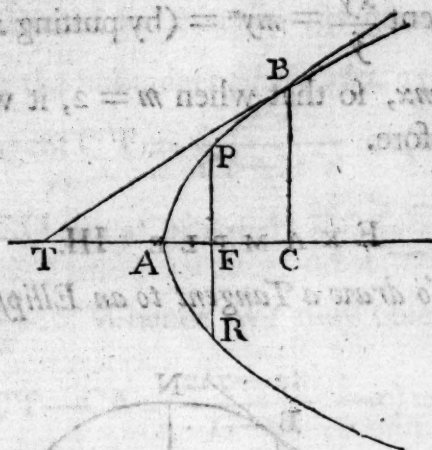
Wherefore if the Distance signified by this Expression be set off from the Point C in the Diameter DA produced, we shall have the Point T to which the Tangent may be drawn.

N. B. The Subtangent above, may be found otherwise, by the Similarity of Triangles only: For the $\angle EBT$ being a right one, the right angled Triangles ECB and BCT will be Similar, and therefore $EC : CB :: BC : CT = \frac{CB^2}{EC}$.

EXAMPLE II.

38. *To draw a Tangent to the Apollonian or common Parabola.*

Suppose F to be the Focus; and PR the Parameter, which put $= a$; also, put the Abscissa $AC = x$, and ordinate $CB = y$; B being the Point to which the Tangent TB is required to be drawn. Then, as is well known, by the Nature of the Curve $PR \times AC = CB^2$, that is, $ax = y^2$. Now by putting both Sides
of



this Equation into Fluxions, we shall have ax

$= 2yy$, therefore $\dot{x} = \frac{2y\dot{y}}{a}$ which substituted for

$\dot{x}$, gives the general Expression for the Sub-

tangent CT , viz. $\frac{\dot{x}y}{y}$ (see *Art.* 35.) $= \frac{2y^2}{a} =$

(by substituting for y^2 , its equal ax above.)

$\frac{2ax}{a} = 2x$. Whence it appears, that the Sub-

tangent CT is double the Abscissa AC , and consequently AT is $= AC$.

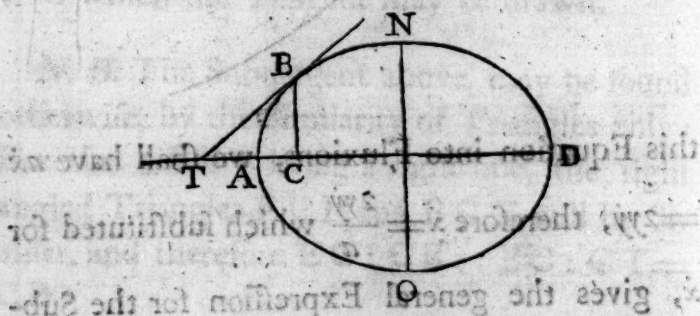
39. But to draw Tangents to all Sorts of Parabolas universally; let the Abscissa $= x$, ordinate $= y$, and the Parameter $= 1$. Then the Equation will be $x = y^2$, and in Fluxions $\dot{x} = 2y\dot{y}$, which substituted for $\dot{x}$ gives the

Sub-

Subtangent $\frac{xy}{y} = my^m =$ (by putting x for y^m its value,) mx , so that when $m = 2$, it will be $= 2x$ as before.

EXAMPLE III.

40. To draw a Tangent to an Ellipsis.



Put the transverse Diameter $AD = t$, Conjugate $NO = c$, Abscissa $AC = x$, and ordinate $CB = y$. Now, by the Nature of the Curve, $[AD] : [NO]^2 :: AC \times CD : [CB]^2$,

i. e. $t : c^2 :: x(t-x) : y^2$, and

by putting each Side of this Equation into

Fluxions we shall have $\frac{c^2}{t} \frac{t-2x}{t} = 2xy$ and

this divided by $\frac{c^2}{t} \frac{t-2x}{t}$ gives $x = \frac{2t^2yy}{t^2 - 2c^2x}$

which

which substituted for x in $\frac{xy}{y}$ the general Expression for the Subtangent (*see Art. 35*), gives the Subtangent $CT = \frac{2t^2y^2}{c^2t - 2c^2x} =$ (by writing

$$\frac{c^2}{t^2} tx - x^2 \text{ for } y^2 \text{ its equal) } \frac{2c^2t^3x - 2c^2t^2x^2}{c^2t^3 - 2c^2t^2x} \\ = \frac{2tx - 2x^2}{t - 2x}.$$

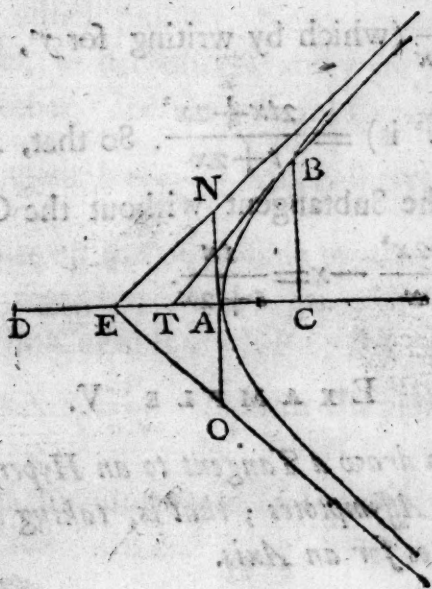
Whence we may observe that

$$AT (= CT - CA = \frac{2tx - 2x^2}{t - 2x} - x) = \frac{tx}{t - 2x}$$

= that Part of the Subtangent which falls without the curve.

EXAMPLE IV.

41. To draw a Tangent to an Hyperbola.



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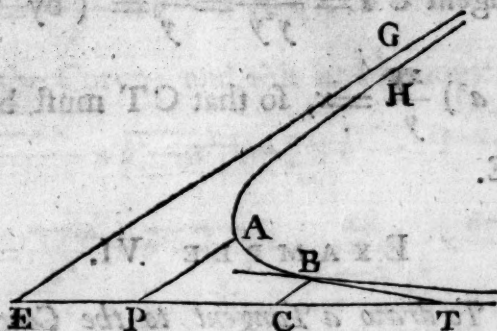
Put

Put the transverse Diameter $DA = t$, Conjugate $NO = c$, Abscissa $AC = x$, and Ordinate $CB = y$. Now, by the Nature of the Curve, as $\overline{DA}^2 : \overline{NO}^2 :: DC \times AC : \overline{CB}^2$ i. e. $t^2 : c^2 :: t + x \times x : \frac{c^2}{t^2} tx + x^2 = y^2$, and by putting both Sides of this Equation of the Curve into Fluxions, we shall have $\frac{c^2}{t^2} t\dot{x} + 2x\dot{x} = 2y\dot{y}$ therefore, by Division, $\dot{x} = \frac{2t^2y\dot{y}}{c^2t + 2c^2x}$ which substituted for $\dot{x}$ in $\frac{\dot{x}y}{\dot{y}}$ or multiplied by $\frac{y}{\dot{y}}$ (*Art.* 35.) gives the Subtangent $CT = \frac{2t^2y^2}{c^2t + 2c^2x}$ (which by writing for y^2 , its equal $\frac{c^2}{t^2} tx + x^2$ is) $= \frac{2tx + 2x^2}{t + 2x}$. So that, AT, that Part of the Subtangent without the Curve, is $= \frac{2tx + 2x^2}{t + 2x} - x = \frac{tx}{t + 2x}$.

EXAMPLE V.

42. To draw a Tangent to an Hyperbola between its Affymptotes; that is, taking one of its Affymptotes for an Axis.

Let



Let EG and ET be the Affymptotes of the Hyperbola HAB, whose Vertex is A, draw AP and BC parallel to the Affymptote EG: then will AP be the Parameter, and equal to PE, which put $= p$; EC an Absciss, which put $= x$; and CB an Ordinate, which put $= y$. Now, because when x increases, y decreases, $\dot{x}$ and $\dot{y}$ are Negative to each other, and the general Expression for the Subtangent is $-\frac{\dot{x}y}{\dot{y}}$ which shews that the

Point T lies on the other Side of the Ordinate CB with regard to E (*Art. 36.*) By the property of the Curve EC: EP :: PA: CB, *i.e.*

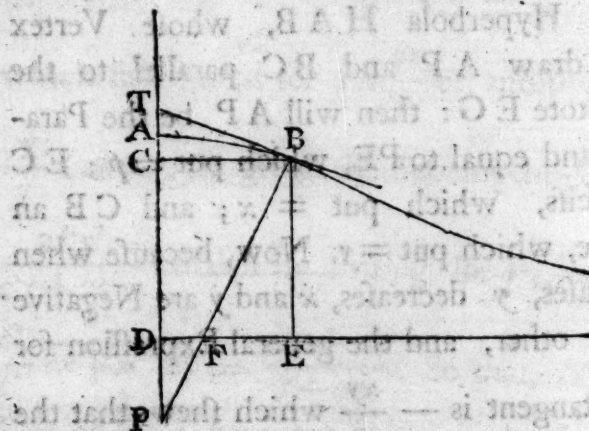
$$x:a::a:y \therefore x = \frac{a^2}{y} \text{ which in Fluxions is } \dot{x} =$$

$$-\frac{a^2\dot{y}}{y^2} \text{ and this substituted for } \dot{x} \text{ in } \frac{\dot{x}y}{\dot{y}} \text{ gives the}$$

Subtangent $CT = \frac{a^2 yy}{y^2 y} = \frac{a^2}{y} =$ (by writing xy for a^2) $\frac{xy}{y} = x$, so that CT must be equal to CE .

EXAMPLE VI.

43. To draw a Tangent to the Conchoid of Nicomedes *.



Let fall the Perpendicular BE on the Asymptote DE and draw BC equal and parallel to ED . Put $PD = a$, $DA = b = FB$, $DC = x = EB$, and $CB = y = DE$. Then by 47 *E. I.* $\sqrt{FB^2 - BE^2} = EF$, *i. e.* $\sqrt{b^2 - x^2} = EF$. But the Δ 's CPB and BEF are Similar, therefore $BE : EF :: PC : CB$, *i. e.* $x : \sqrt{b^2 - x^2} :: a$
 $+x$

$+x : y = \frac{a+x}{x} \times \sqrt{b^2-x^2}^{\frac{1}{2}}$ which is the Equa-

tion of the Curve; and this in Fluxions is $\dot{y} =$

$$\frac{x\dot{x} - a - x\dot{x}}{x^2} \times \sqrt{b^2-x^2}^{\frac{1}{2}} - \frac{1}{2} \times \sqrt{b^2-x^2}^{-\frac{1}{2}} 2x\dot{x} \times$$

$$\frac{a+x}{x} = \frac{-a\dot{x}}{x^2} \times \sqrt{b^2-x^2}^{\frac{1}{2}} - \frac{x\dot{x}}{\sqrt{b^2-x^2}^{\frac{1}{2}}} \times \frac{a+x}{x} =$$

$$\frac{-ab^2\dot{x} - x^3\dot{x}}{x^2 \times \sqrt{b^2-x^2}^{\frac{1}{2}}} \text{ which substituted for } \dot{y} \text{ in } -\frac{\dot{x}y}{\dot{y}},$$

the general Expression for the Subtangent when $\dot{x}$ and $\dot{y}$ are Negative to each other (*Art. 36.*) gives the Subtangent CT =

$$\frac{yx^2 \times \sqrt{b^2-x^2}^{\frac{1}{2}}}{ab^2+x^3} \text{ (which by substituting for } y \text{ its}$$

equal in the above Equation of the Curve, is) =

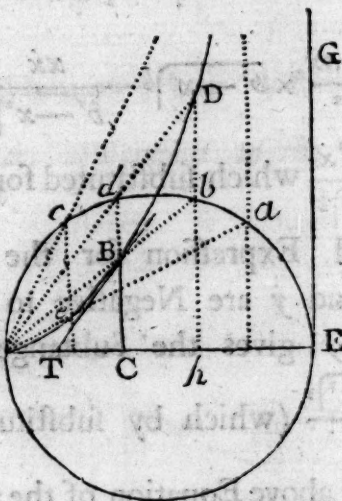
$$\frac{a+x \times \sqrt{b^2-x^2} \times x^2}{ab^2x+x^3} = \frac{a+x \times \sqrt{b^2-x^2}}{ab^2+x^3}.$$

* This Curve is thus generated; from a fixt Point P, which is called the Pole of the Conchoid, let any Number of right Lines PA, PB, be drawn cutting the right Line DE, which is an Affymptote to the Curve; and let the Distances DA, FB, be made equal to each other, and a Line drawn thro' the Points A, B; then will this Line be a Curve, called by its Inventor *Nicomedes*, a Conchoid.

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EXAMPLE VII.

Diocles*.



Let ABD be the Cissoid, whose generating Circle is AFE and Asymptote EG. Put the Diameter $AE=a$, Absciss $AC=x$, and Ordinate $CB=y$. Now by the Nature of the Curve $dC : CA :: AC : CB$ (For since by the Generation of the Curve, the Arches Eb and Ad must be equal, therefore $bb=dC$ and $bE=CA$; and because the Δ 's bbA and BCA are alike, therefore $Ab : bb :: AC : CB$; but by the Property of Circles $Ab : bb :: bb : bE :: bb : bE :: AC : CB$, i. e. $dC : CA$

$\therefore CA :: AC : CB,$ or, because $CE = a - x$,
 and $AC \times CE^{\frac{1}{2}} = \overline{ax - x^2}^{\frac{1}{2}} = Cd$, as $\overline{ax - x^2}^{\frac{1}{2}}$
 $: x :: x : y \therefore x^2 = y \times \overline{ax - x^2}^{\frac{1}{2}}$ and by squar-
 ing both Sides of this Equation $x^4 = axy^2 -$
 x^2y^2 , and this divided by x gives $x^3 = ay^2 - xy^2$.
 In Fluxions it is $3x^2\dot{x} = 2ay\dot{y} - \dot{x}y^2 - 2xy\dot{y}$,
 then transpose and divide and it will be $\dot{x} =$
 $\frac{2ay\dot{y} - 2xy\dot{y}}{3x^2 + y^2}$ which substituted for $\dot{x}$ gives the

Subtangent $CT (= \frac{\dot{x}y}{\dot{y}}, \text{ Art. 35.}) =$

$\frac{2ay^2 - 2xy^2}{3x^2 + y^2} =$ (by substituting for y^2 , its equal

$\frac{x^3}{a - x} \Big) \frac{2ax - 2x^2}{3a - 2x}$. Whence we may observe

that AT , the difference between the Abscissa

AC and Subtangent TC , is $= \frac{ax}{3a - 2x}$.

* This Curve is thus generated. Let $A b E$
 be a Circle, whose Diameter is $A E$; and
 make any Arches $E a$, $E b$, and $A c$, $A d$, e-
 qual to one another; and thro' the Points a ,
 b , d , c , let right Lines be drawn Perpendicular
 to $A E$, and transverse Lines from the Point
 A ; then, from A , thro' the Points of Inter-
 section e , B , D , &c. draw a Line $A e B D$ &c.
 and

DOCTRINE of FLUXIONS. 57

Let $Gg = z'$, $gn = s'$, and $nG = Cc = \dot{x}$; then, because EG is Perpendicular to Gg , the $\angle$'s gGn and EGC are equal, and therefore the right angled Δ s gnG and GCE are alike; and consequently $gn : nG :: EC : CG$, that

$$\text{is, } s' : x' :: a - x : s \therefore s' = \frac{a - x}{s} x' \text{ or } \dot{s} =$$

$$\frac{a - x}{s} \dot{x}, \text{ and again, } gG : Gn :: EG : GC,$$

$$\text{that is, } z' : x' :: a : s \therefore z' = \frac{a}{s} x' \text{ or } \dot{z} = \frac{a}{s} \dot{x}$$

Now by substituting $\frac{a - x}{s} \dot{x}$ and $\frac{a}{s} \dot{x}$ for $\dot{s}$ and $\dot{z}$

in the above Fluxion of the Equation of the

$$\text{Curve, we have } \dot{y} = \frac{a - x}{s} \dot{x} + \frac{a}{s} \dot{x} = \frac{2a - x}{s} \dot{x}$$

therefore the Subtangent $CT (= \frac{\dot{x}y}{\dot{y}} \text{ Art. 35.})$

$$= \frac{sy}{2a - x}. \text{ Whence we may observe, by An-}$$

nalogy $2a - x : s :: y : CT$, *i. e.* $FC : CG :: BC : CT$; but by the Property of Circles $FC : CG :: GC : CA \therefore GC : CA :: BC : CT$; consequently the Δ s GCA , and BCT are alike, and BT is parallel to GA .

* This Curve is thus generated. Let a Circle, or Wheel, roll along upon a right Line,

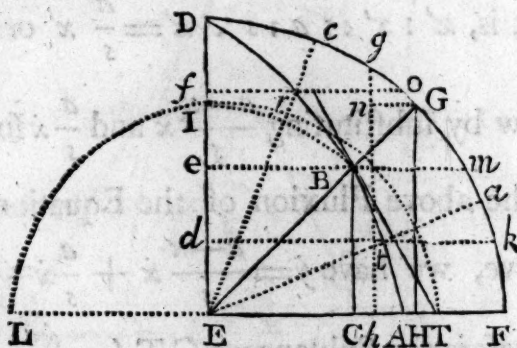
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until it perform one Revolution, that is, until it measure out a right Line equal to its Circumference; then that Point in the Circle which first touched the right Line, will describe the Curve called a Cycloid.

EXAMPLE IX.

46. To draw a Tangent to the Quadratrix of Dinostratus *.



Put EF , the Radius of the generating Quadrant $= a$; EA the Base of the Quadratrix $= b$; Quadrantal Arch $FD = c$; Absciss $AC = x$; Ordinate $CB = y$; Arch $FG = z$; and the Sine $GH = s$. Now, by the Nature of the Curve, as Arch FD : Rad. $DE ::$ Arch FG : Ordinate BC (which is evident from the generation of the Curve,) that is, $c : a :: z : y$

$\therefore az = cy$ and $z = \frac{cy}{a}$ in Fluxions $\dot{z} = \frac{c\dot{y}}{a}$. By

Sim.

Sim. $\Delta s BC : CE :: GH : HE$, *i. e.* $y : b -$

$x :: s : \frac{sb - sx}{y} = HE$. Let gb be conceived

indefinitely near and parallel to GH and Gn equal and parallel to Hb , *i. e.* let $ng = s'$ and $gG = z'$, then will the $\Delta s EHG$ and gnG be alike, and therefore $HE : EG :: ng : gG$,

i. e. $\frac{sb - sx}{y} : a :: s' : z' = \frac{as'y}{sb - sx}$ or (*Art. 3.*)

$\dot{z} = \frac{asy}{sb - sx}$. Hence we have $\frac{asy}{sb - sx} = \frac{cy}{a}$

which gives $\dot{s} = \frac{sbcy - sxcy}{a^2y}$. Now by 47 *E. 1.*

$\overline{EC^2 + CB^2}^{\frac{1}{2}} = BE$, *i. e.* $\overline{b-x^2 + y^2}^{\frac{1}{2}} = BE$,
and by Sim. $\Delta s EB : BC :: EG : GH$, *i. e.*

$\overline{b-x^2 + y^2}^{\frac{1}{2}} : y :: a : s = \frac{-ay}{\overline{b-x^2 + y^2}^{\frac{1}{2}}}$ or $s =$

$\frac{ay}{b^2 - 2bx + x^2 + y^2}^{\frac{1}{2}}$ in Fluxions $\dot{s} =$

$\frac{ay \times \overline{b-x^2 + y^2}^{\frac{1}{2}} + \frac{bxay - x\dot{x}ay - y\dot{y}ay}{\overline{b-x^2 + y^2}^{\frac{1}{2}}}}{\overline{b-x^2 + y^2}} =$

$\frac{ay \times \overline{b-x^2 + y^2}^{\frac{1}{2}} + bxay - x\dot{x}ay}{\overline{b-x^2 + y^2}^{\frac{3}{2}}} =$ (because $\dot{s}$ by the

above is $=$) $\frac{sbcy - sxcy}{a^2y}$, and this Equation, by

Multiplication and Transposition, gives $a^3by^2\dot{x} - a^3y^2x\dot{x} = bcsy - csxy \times \sqrt{b-x}^2 + y^2^{\frac{3}{2}} - a^3yy \times \sqrt{b-x}^2$, and this divided by $b-x$ gives $a^3y^2\dot{x} = csy \times \sqrt{b-x}^2 + y^2^{\frac{3}{2}} - a^3yy \times \sqrt{b-x}$ (or by substituting for s its above value $\frac{ay}{\sqrt{b-x}^2 + y^2^{\frac{3}{2}}}$, $= acyy$

$\times \sqrt{b-x}^2 + y^2 - a^3yy \times \sqrt{b-x}$: But, by a Property of the Curve, as $FD : DE :: EF : EA$ (for when the Arch FG is indefinitely small, it will coincide with, or be equal to its Sine GH , and one may be taken for the other; and then EA may be taken for EC , and EF for EH ; and therefore then, by the above Nature of the Curve, as Quadrantal Arch FD : Rad. $DE ::$ indefinitely small Sine GH : indef. small Ordinate BC ; but by Sim. Δ $GH : BC :: (HE, i. e.) EF : (CE, i. e.) EA$ conseq. Arch FD : Rad. $DE :: EF : EA$, *i. e.* as $c : a :: a : b \therefore bc = a^2$, and by substituting bc for a^2 we have $abcy^2\dot{x} = acyy \times \sqrt{b-x}^2 + y^2 - abcyy \times \sqrt{b-x}$ and this Equation divided by acy gives $by\dot{x} = y \times \sqrt{b-x}^2 + y^2 - by \sqrt{b-x} \therefore \dot{x} = y \frac{\sqrt{b-x}^2 + y^2 - b \times \sqrt{b-x}}{by}$ which

substituted for $\dot{x}$ gives the Subtangent $CT (=$
 $\dot{x}y$

$$\frac{\dot{x}y}{y} \text{ Art. 35.} = \frac{\overline{b-x_1^2+y^2}}{b} - \overline{b-x} = \frac{\overline{EB}^2}{EA}$$

EC, which gives the following geometrical Construction, *viz.* if B be the Point to which the Tangent is to be drawn, make $E I = E B$, and $E L = E A$, and let the Semicircle LIT be drawn thro' the Points L and I , then T is the Point thro' which the Tangent must pass. For by the Property of Circles $LE \times ET = \overline{EI}^2 = (\text{by Construction}) \overline{EB}^2$, *i. e.* $EA \times \overline{EC+CT} = EB^2 \therefore \frac{\overline{EB}^2}{EA} = EC + CT$ and $\frac{\overline{EB}^2}{EA} - EC = CT$.

* This Curve is thus generated. Let EFD be a Quadrant of a Circle, whose Radius ED divide into any Number of equal Parts (as Ed , de , ef , fD) and from the Points of Division (d , e , f ,) draw right Lines (dk , em , fo ,) parallel to each other and to the Radius EF ; also divide the Arch FD into the same Number of equal Parts (Fa , aG , Gc , cD ,) as you do the Radius ED , and to these Points of Division (a , G , c ,) draw Radii (Ea , EG , Ec ,) from the Point or Center E ; then a Line drawn from D thro' the Points of Intersection (r , B , t , &c.) will

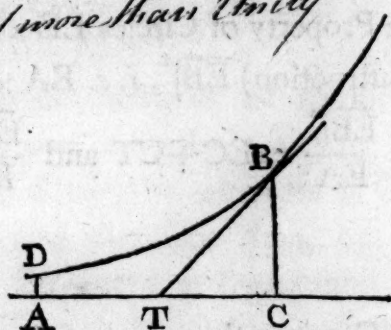
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will be the Curve called a Quadratrix; the
Invention of which is imputed to *Dinostratus*.

EXAMPLE X.

47. To draw a Tangent to the exponential
Curve DB, whose Equation (putting $AC=x$,
 $CB=y$, and $a=a$ given Quantity) is $a^x=y$.

(not add more than unity)



PUT A = the hyperbolic Logarithm of a ,
and Y = the hyperbolic Logarithm of y ; then
by the Nature of Logarithms $x A = Y$, in
Fluxions $\dot{x} A = \dot{Y} =$ (see Art. 14.) $\frac{\dot{y}}{y}$ which

divided by A gives $\dot{x} = \frac{\dot{y}}{A y}$ and this substituted

for $\dot{x}$ gives the Subtangent $CT (= \frac{\dot{x} y}{\dot{y}} \text{ Art.}$

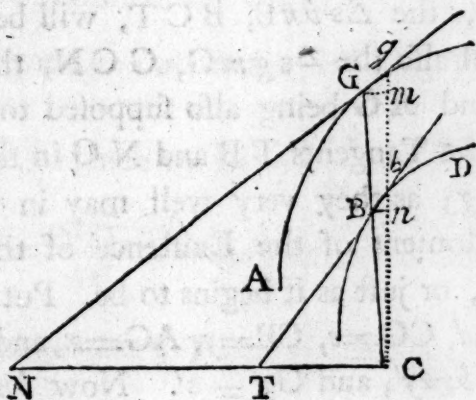
35.) $= \frac{y}{A y} = \frac{1}{A}$. Whence we may observe

that,

that, the Subtangent being an invariable Quantity, the Curve DB is the Logarithmic Curve, (see Art. 70.)

48. The Examples already given being only of those Curves, which are referred to an Axis, or whose Ordinates are parallel to one another; we shall now give a few Examples of Spirals; or those whose Ordinates all issue from one and the same Point: Where note the general Expression for the Subtangent will be $\frac{\dot{x}y}{\dot{y}}$ as before, as will be proved in the next Example.

EXAMPLE XI.



49. Let AG be any Curve whose Ordinates all issue from the fixt Point C, and suppose its Tangent GN, Subtangent NC, and Ordinate CG

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CG to be given; and let there be another Curve, as BD, whose Ordinates also issue from the same Point C, and of such a Nature, that AG shall always be to CB as a to b, 'tis required to draw the Tangent BT.

N. B. The Subtangent and Ordinate to all Spirals, are always perpendicular to each other.

SUPPOSE Cg indefinitely near to CG , that is, let the $\angle G C g$ be supposed indefinitely small; and with the Radii CB, CG , let the indefinitely small concentric Arches Bn, Gm , be described; which being considered as coinciding with Tangents to the Points n and m , or as indefinitely small right Lines Perpendicular to Cg , the $\Delta s b n B, B C T$, will be Similar, as will also the $\Delta s g m G, G C N$, the Curves BD and AG being also supposed to coincide with the Tangents TB and NG in the Points b and g ; as they very well may in the very first Moment of the Existence of the Angle $G C g$, or just as it begins to be. Put $GN=c$, $NC=d$, $CG=e$, $CB=y$, $AG=z$, and $Bn=x'$; then $nb=y'$, and $Gg=z'$. Now $bn : nB ::$
 $BC : CT$, *i. e.* $y' : x' :: y : \frac{x'y}{y'} = CT$, or $y : x$

$:: y$

$\therefore y : \frac{\dot{x}y}{\dot{y}} = CT$, which is a general Expression for the Subtangent of Spirals, or Curves referr'd to a fixt or central Point, and is the same as that before found for those Curves which are referred to an Axis, *Art.* 35.

From the similar Sectors or Δs CBn and CGm we have $CB : Bn :: CG : Gm$, *i. e.*

$y : x' :: e : \frac{ex'}{y} = Gm$. But by the sim. Δs

gmG and GCN we have $GN : NC :: gG$

$: Gm$, *i. e.* $c : d :: z' : \frac{dz'}{c} = Gm$; there-

fore $\frac{dz'}{c} = \frac{ex'}{y}$ and $z' = \frac{cex'}{dy}$ or $\dot{z} = \frac{ce\dot{x}}{dy}$.

Now by the Nature of the Curve $AG : CB ::$

$a : b$, that is, $z : y :: a : b$ or $z = \frac{ay}{b}$ in Flux-

ions $\dot{z} = \frac{a\dot{y}}{b}$. Hence we have $\frac{ce\dot{x}}{dy} = \frac{a\dot{y}}{b}$ and

therefore $\dot{x} = \frac{ady\dot{y}}{bce}$ which substituted for $\dot{x}$

gives the Subtangent $CT (= \frac{\dot{x}y}{\dot{y}}) = \frac{ady^2}{bce}$.

duc'd gives $\dot{x} = \frac{ay\dot{y}}{b^2}$ which substituted for $\dot{x}$

gives the Subtangent CT ($= \frac{\dot{x}y}{\dot{y}}$, *Art.* 48.) =

$\frac{ay^2}{b^2} = \frac{yz}{b}$ by substituting bz for its Value ay ;

which gives the following Construction, *viz.* with the Ordinate CB, as a Radius describe the circular Arch BD, and make CT perpendicular to CG and equal to the Arch BD; then will T be the Point thro' which the Tangent must pass: For then the Sectors CGA and CBD will be Similar and conseq. CG :

GA :: CB : BD, *i. e.* $b : z :: y : BD = \frac{yz}{b}$
= CT.

N. B. The above Expression for the Subtangent may be inferred from that in *Art.* 49. for if in that Article, the Curve AG be an Arch of a Circle, the Tangent GN and Subtangent NC will both be infinite (since the Tangent to a Circle is perpendicular to the Radius, and conseq. CG being the Radius, GN and NC will be parallel;) and a denote the Circumference, and b the Radius of that Circle; and therefore b and e will be equal; and substituting b for e and striking c and d out
of

of $\frac{ady^2}{bce}$, the Expression for the Subtangent ;

we shall have $\frac{ay^2}{b^2} =$ the Subtangent as above.

* This Curve is thus generated. With the Radius CA, let the Circle AIGA be described with an equable Motion, or, the Point A describe equal Arches in equal Times ; and at the same Moment of Time that the Point A begins to generate the Circle, let another Point be conceived to begin to move along the Radius CA from C towards A, and to pass over it with an uniform Motion, and such Velocity that it may arrive at A at the very same Moment of Time that the Radius CA shall have described the Circle, or come to be in its first Situation : Then will the Point moving along the Radius CA, generate, or describe, the Curve CBbA called a Spiral, which Name was given to it by its Inventor, *Archimedes*.

C H A P. V.

Of finding the Points of Inflection or contrary Flexure in CURVES.

HITHERTO we have had no Occasion to make use of second Fluxions, and therefore have not treated of them: But here we shall say something concerning them, as they will be used in this, and the following Chapter.

51. IN all fluxional Equations, it is proper that some one of the variable Quantities be supposed to encrease uniformly, with which the others may be always compared; which Quantity will therefore have no second Fluxion, that is, the first Fluxion of it will be invariable. Now the Reason of this is plain; for if we only know in general, that two or more Bodies, are carried over unequal Spaces in unequal Times, no Inference can from thence be drawn, with regard to the Equality or Inequality of their Velocities; but if the Spaces pass'd over, are compared with the Times in which the Bodies were moving, we shall then obtain the Ratio of the Velocities of the moving Bodies: Now in Order to this Comparison, there must necessarily be some Standard which is the common

common Measure of the Times the Bodies were in Motion ; the same as, in the Comparison of two or more Magnitudes, there must be always some Quantity, which is the common Measure of all. And, in Curves, when we make the Fluxion of any flowing Quantity, as the Absciss for Instance, invariable ; it is, in Effect, no more than taking equal Portions of it, that we may thereby determine the Fluxions of the other variable Quantities, or the Proportions with which they flow.

52. In finding the second Fluxion of any Equation, all the variable Quantities and Fluxions, must be considered as distinct Fluents, and fluxed accordingly, by the Rules laid down in *Chap. 2*. Thus the Fluxion of the Equation $zy = ax$, by *Art. 9*. will be $\dot{z}y + z\dot{y} = a\dot{x}$, but as we may assume either $\dot{x}$ or $\dot{y}$ as invariable, by making $\dot{x}$ so, it will be $\dot{z}y + z\dot{y} = 0$, because $\dot{x}$, and consequently $a\dot{x}$, is $= 0$, or if we make $\dot{y}$ invariable, it will be $\dot{z}y = a\dot{x}$, because then $\dot{y}$ and consequently $z\dot{y}$ will be $= 0$.

Again the Fluxion of $\frac{a\dot{z}x}{\dot{x}^2} + \frac{y\dot{y}}{a} = \dot{y}$, if we make $\dot{y}$ invariable or $\dot{y} = 0$, will be

$$\frac{a\ddot{z}x\dot{x}^2 + a\dot{z}\dot{x}^3 - 2\dot{x}\ddot{x}a\dot{z}x}{\dot{x}^4} + \frac{\dot{y}^2}{a} = \ddot{y} = 0, \text{ that is,}$$

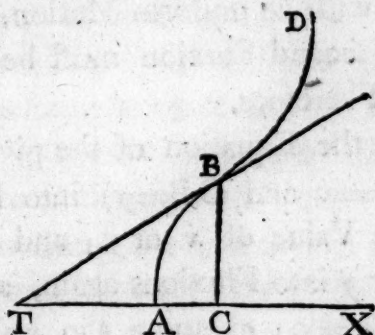
$$a\ddot{z}x\dot{x} +$$

$$\frac{a\ddot{x}x\dot{x} + a\dot{x}\ddot{x}^2 - 2\ddot{x}a\dot{x}\dot{x}}{\dot{x}^3} + \frac{\dot{y}^2}{a} = 0. \quad \text{The third or}$$

fourth Fluxions, &c. are found in the same Manner, due regard being had to such Fluxions as are supposed invariable.

53. When a Curve from being Concave becomes Convex towards its Axis, or from being Convex becomes Concave, that Point where the Change is made, or that which separates the Convex from the Concave Part, is called the Point of Inflection, or contrary Flexure; or, the Point of Inflection is that, to which a Tangent being drawn, cuts the Curve.

54. In the Curve ABD , suppose B to be the Point of Inflection; then, because AB is



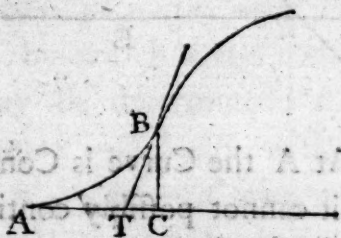
Concave and BD Convex towards the Axis AX , is it evident, that the indefinitely small Increment of the Curve at the said Point B , will be neither Concave nor Convex, that is, will be a right Line; and because, during this Increment

crement the Tangent at B will exactly coincide with the Curve, or the Ordinate will flow on each Side of the Point B with an equable or uniform Motion, the Absciss being supposed to flow so too, it follows, that the Ordinate will by flowing along that Increment, have no variation of Increase, or its second Fluxion will be $=0$. Or, because when the Absciss flows on with an uniform Motion, the Ordinate flows with a retarded Motion, when the Curve is Concave towards the Absciss, and with an accelerated Motion when it is Convex, therefore at the Point of Inflection, where the Curve is neither Concave nor Convex the Ordinate must flow, with neither a retarded nor accelerated, but, with an uniform Motion, and consequently its second Fluxion must be equal to Nothing. Therefore,

55. PUT the Equation of the given Curve (where $AC=x$ and $CB=y$) into Fluxions, and find the Value of $\dot{x}$ or $\dot{y}$; and put this Value of $\dot{x}$ or $\dot{y}$ into Fluxions again, and make both $\ddot{x}$ and $\ddot{y} = 0$; expunge the rest of the fluxional Quantities, by the help of the Fluxion of the Equation of the Curve; and you will have the Point of Inflection sought, determined.

56. *Note,* When the Curve is first Concave towards

towards the Axis, as in the last Figure, it is evident, if B be the Point of Inflection, and BT a Tangent to it, that AT the Difference between the Subtangent and Absciss will be a *Maximum*: and when the Curve is first Convex towards the Axis, as in the annexed Figure, it is evident, that AT the Difference be-



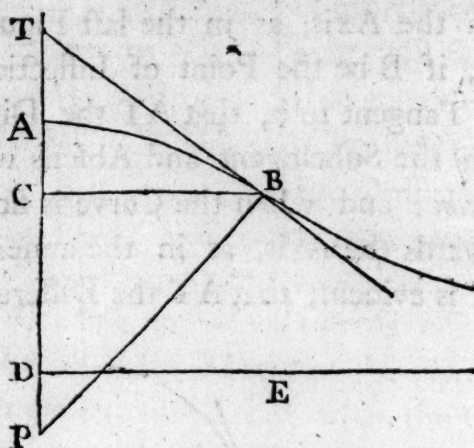
tween the Absciss and Subtangent will be also a *Maximum*. Wherefore the Point of Inflection may be determined without the help of second Fluxions, by first finding a definitive Expression for the Subtangent, by Chap. 4. and the Difference between that and the Absciss, and then making the Fluxion of this Difference equal to Nothing. This we shall illustrate in the following Example.

EXAMPLE I.

57. To find the Point of Inflection in the Conchoid of Nicomedes, or, the Point where it begins to be Convex towards the Assymptote DE.

L

N. B.



N. B. At A the Curve is Concave towards DE; but it cannot possibly continue so long; for, if it did, the farther it proceeded the more it would incline or tend towards the said Line, and so intersect it; whereas in fact, the farther it proceeds, the less it tends towards the said Line: Consequently the Curve must have a Point of Inflection.

Put $PD=a$, $DA=b$, $DC=x$, and $CB=y$; then by *Art. 43.* the Fluxion of the Equation of the Curve will be $y = \frac{-ab^2\dot{x} - x^3\dot{x}}{x^2 \times b^2 - x^2}^{\frac{1}{2}}$

Suppose B to be the Point of Inflection, then the Fluxion of this again, (because x being supposed to flow with an equable Motion, the second Fluxions of x and $y=0$ will be $0 = -3x^4$

$$\frac{-3x^4\dot{x}^2 \times \sqrt{b^2-x^2}^{\frac{1}{2}} - 2x\dot{x} \times \sqrt{b^2-x^2}^{\frac{1}{2}} - \frac{x^3\dot{x}}{\sqrt{b^2-x^2}^{\frac{1}{2}}}}{x^4}$$

$$\frac{-\frac{ab^2\dot{x}-x^3\dot{x}}{x\sqrt{b^2-x^2}}}{x^4 \times \sqrt{b^2-x^2}^{\frac{1}{2}}}, i. e. 0 = \frac{2ab^4x - 3ab^2x^3 - b^2x^4}{x^4 \times \sqrt{b^2-x^2}^{\frac{1}{2}}} \dot{x}^2$$

which multiplied by $x^4\sqrt{b^2-x^2}^{\frac{1}{2}}$ and divided by $b^2x\dot{x}^2$ gives $0 = 2ab^2 - 3ax^2 - x^3$, or $x^3 + 3ax^2 - 2ab^2 = 0$, by which x and consequently the Point B may be determined: And if $a=b$ it will be $x^3 + 3ax^2 - 2a^3 = 0$, which divided by $x+a$ gives $x^2 + 2ax - 2a^2 = 0$, or $x^2 + 2ax = 2a^2$, and by solving the Quadratic $x = \sqrt{3a^2}^{\frac{1}{2}} - a$.

Or, because the Subtangent CT (*Art.* 43.) is $= \frac{a+x \times \sqrt{b^2-x^2}}{ab^2+x^3}$, and AC is $= b-x$;

therefore AT ($= CT - CA$) $= \frac{a+x \times \sqrt{b^2-x^2}}{ab^2+x^3}$

$- b+x = a$ Maximum (*Art.* 56.) which thrown into Fluxions is

$$\frac{ab^2\dot{x} - 3ax^2\dot{x} + 2b^2x\dot{x} - 4x^3\dot{x} \times \sqrt{ab^2+x^3}}{(ab^2+x^3)^2} \text{ or}$$

$$-\frac{3x^2\dot{x} \times \sqrt{ab^2x - ax^3 + b^2x^2 - x^4}}{a^2b^4 + 2ab^2x^3 + x^6} + \dot{x} = 0,$$

then multiply by $a^2b^4 + 2ab^2x^3 + x^6$ and strike out the contradictory Terms, and it will be

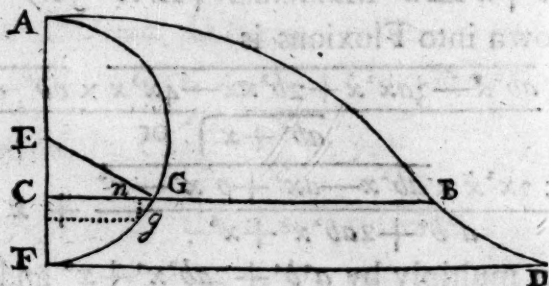
$$2a^2b^4\dot{x} - 4ab^2x^3\dot{x} - 3a^2b^2x\dot{x} + 2ab^2x\dot{x} - b^2x\dot{x} = 0,$$

$=0$, which divided by $-ab^2x - b^2xx$ gives $x^3 + 3ax^2 - 2ab^2 = 0$, as before.

We may observe here the strict Agreement in these two Methods of Solution; since, tho' the Premisses are widely different, yet the Conclusions are exactly the same: And indeed, in all fluxional Operations, where the Thing sought may be obtained by different Methods of reasoning, these Methods do never serve to give us different Expressions of one and the same Value, but always the very same Expression; unless in some Cases where there is any Ambiguity in the algebraic Operation.

EXAMPLE II.

58. To find the Point of Inflection in the inflected Cycloid *, the Circumference of whose generating Circle is to its Base as 4 to 5.



PUT the generating Semicircle $A G F = a$,
Base $F D = b$, Radius $E A$ or $E F = r$, Absciss
A C

$AC=x$, Ordinate $CB=y$, Arch $AG=z$, and Sine $GC=s$; and suppose B to be the Point of Inflection sought. Now, by the Nature of the Curve $a:b::z:(GB) y-s::ay-as=bz$,

and $y=s+\frac{bz}{a}$ and the Fluxion of this Equa-

tion is $\dot{y}=\dot{s}+\frac{b\dot{z}}{a}$. Let the fluxional Triangle

Ggn and the Radius EG be drawn; then will the Triangles ECG and Gng be alike, (for Gg being considered as an indefinitely small right Line coinciding with the Tangent to the Point G , the $\angle E G g$ will be a right Angle, and therefore $=\angle CEG+\angle EGC$, and the $\angle EGC$ being common, therefore $\angle CEG=\angle n G g$ and conseq. the $\angle s GCE$ and Gng being right, the $\angle EGC=\angle Ggn$;) therefore, as $nG:ng::EC:CG$, or, $\dot{x}:\dot{s}::x-r:s$ being Negative to each other, as

$-\dot{s}:\dot{x}::x-r:s::\dot{s}=\frac{r-x\times\dot{x}}{s}$, but by the

Property of the Circle $GC=\overline{AC\times CF}^{\frac{1}{2}}$, i. e.

$s=\overline{2rx-x^2}^{\frac{1}{2}}$ which substituted for s gives $\dot{s}=\frac{r-x\times\dot{x}}{\overline{2rx-x^2}^{\frac{1}{2}}}$. By 47 E. 1. $\dot{z}=\overline{x^2+s^2}^{\frac{1}{2}}$ (i. e. by

substituting for s its Value) $=\frac{r\dot{x}}{\overline{2rx-x^2}^{\frac{1}{2}}}$.

Whence

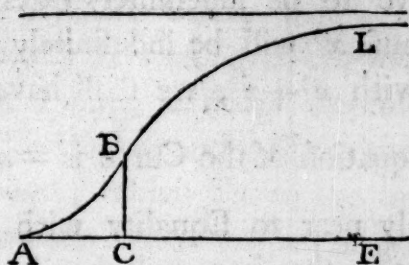
Whence $y (= \dot{s} + \frac{b\ddot{z}}{a}) = \frac{r-x \times \dot{x}}{2rx-x^2|^{\frac{1}{2}}} + \frac{br\dot{x}}{a \times 2rx-x^2|^{\frac{1}{2}}} = \frac{ar+br-ax}{a \times 2rx-x^2|^{\frac{1}{2}}} \dot{x}$. And this put into Fluxions, supposing x to flow with an uniform Motion, or $\ddot{x}$ and $\ddot{y}=0$, (*Art.* 54.) is $0 = \frac{brx-ar^2-br^2}{a \times 2ax-x^2|^{\frac{1}{2}}} \dot{x}^2$, consequently $brx-ar^2-br^2=0$, and $brx=ar^2+br^2 \therefore x = \frac{a+b}{b}r$, and $EC (= x-r) = \frac{a}{b}r$, which gives the Point C; from which a perpendicular Ordinate being drawn, will fall on the Point of Inflection: And this is an universal Theorem for all inflected Cycloids, when $\frac{a}{b}$ expresses the Ratio of the Circumference of the generating Circle to the Base of the Cycloid.

* See how this Curve may be generated, *Chap.* 14. *Quest.* 7.

EXAMPLE III.

59. To find the Point of Inflection B in the Curve ABD, whose Equation (putting AC= x , CB= y , and a = a given Quantity) is $ax^2=a^2y+x^2y$.

THE



THE Equation of the Curve put into Fluxions is $2ax\dot{x} = a^2\dot{y} + 2x\dot{x}y + x^2\ddot{y}$, therefore $y = \frac{2ax\dot{x} - 2x\dot{x}y}{a^2 + x^2}$ (*i. e.* by substituting $\frac{ax^2}{a^2 + x^2}$ for its

Value y) $= \frac{2a^3x\dot{x}}{a^2 + x^2}^2$, and the Fluxion of this

again, making $\ddot{x}$ and $\ddot{y} = 0$, is $0 = \frac{2a^3\dot{x}^2 \times a^2 + x^2 - 4a^2x\dot{x} + 4x^3\ddot{x} \times 2a^3x\dot{x}}{a^2 + x^2}^4$, theref.

$$2a^3\dot{x}^2 \times a^2 + x^2 - 4a^2x\dot{x} + 4x^3\ddot{x} \times 2a^3x\dot{x} = 0$$

and by dividing by $2a^3\dot{x}$, we shall have $\dot{x} \times$

$$a^2 + x^2 - 4a^2x\dot{x} = 0$$

therefore $4x^4\dot{x} +$

$$4a^2x^2\dot{x} = \dot{x} \times a^2 + x^2$$

and this Equation divided by $x^2\dot{x} + a^2\dot{x}$ gives $4x^2 = a^2 + x^2$ by transposition $3x^2 = a^2 \therefore x^2 = \frac{1}{3}a^2$ and $x = a\sqrt{\frac{1}{3}}$ and

if this be substituted for x in the Equation of the Curve, we shall have y or the Ordinate at the Point of Inflection $= \frac{1}{4}a$.

If it were required to find the Asymptote of this Curve, we need only suppose the Absciss

and

and Curve to be indefinitely extended ; and then because x^2 will be indefinitely near to Equality with $a^2 + x^2$, we shall have y , which by the Equation of the Curve is $= a \times \frac{x^2}{a^2 + x^2}$, indefinitely near to Equality with the given right Line a , that is y , will then be $= a$ less a Quantity indefinitely small : Wherefore if a right Line, to the Distance of the given right Line a , be drawn parallel to the Diameter AE it will be the Assymptote required.

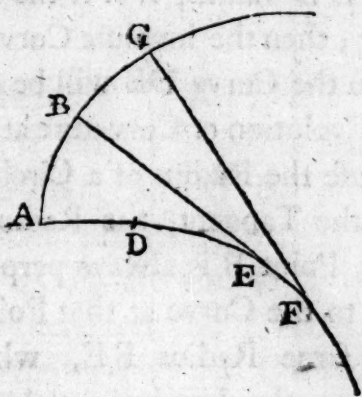
C H A P. VI.

Of finding the Radius of Curvature in Curves.

60. **A**S the Curvature or Convexity of all Curves, but Circles, varies in every Point ; therefore, if Circles are drawn to coincide with the given Curve in any Number of Points, or, which is the same, so that the Tangents to the Circles may coincide with the Tangents to the Curve, the Radii of these Circles will be different : And the finding of these Radii is the Business of this Chapter.

And because all Curves are formed, or may be formed, or generated, by the Evolution or winding off of some other Curves ; therefore the

the Centers of these Circles which coincide; and consequently have an equal Curvature with the different Points, or rather Increments, of these Curves, will be continually in the Curves to be unwound; which Curves are called the Evolutes, and the others formed or generated, or conceived to be generated, by their Evolution, are called the Involute.



61. To make this plainer: Let DEF be any Curve, round which conceive a Thread to be wound and extended beyond the Curve D in a right Line to A: Let this Thread be evolved, or wound off, from the Curve DE, so that it be continually stretched at its full Length as it leaves the Curve; then will the Point A generate, or describe the involute Curve ABG: and AD, BE, GF, will be the Radii of Curvature at the Points A, B, G, M respectively,

respectively. From this Definition, we may draw these following

COROLLARIES.

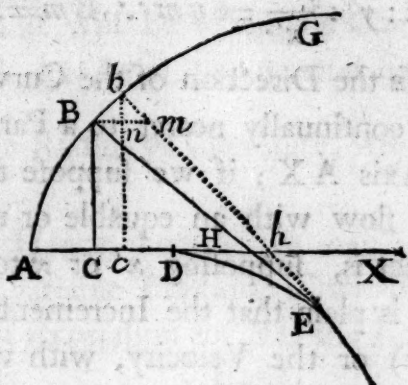
1. The Radius of Evolution or Curvature BE, will be always equal to the Length of the Curve DE, and the right Line AD; and consequently, if the vertical Distance or shortest Radius AD vanish, *i. e.* if the Radius at A be nothing; then the involute Curve will begin at D and so the Curve DE will be equal to the Radius of Evolution or Curvature at the Point B.

2. Because the Radius of a Circle is perpendicular to the Tangent, the Radius of Curvature at any Point B is always perpendicular to a Tangent to the Curve at that Point.

3. The same Radius BE, which is perpendicular to the Involute at the Point B, is also a Tangent to the Evolute at the Point E.

62. *To deduce a general Expression for BE, the Radius of Evolution or Curvature, for any Point B of the involute Curve ABG, whose Axis is AX, and Evolute DE.*

Put the Absciss $AC = x$, Ordinate $CB = y$; suppose bE indefinitely near to BE , bc indefinitely near and parallel to BC , and Bm parallel to AX , *i. e.* let Cc or $Bn = x'$, and $nb = y'$;



$nb = y'$; then by 47 E. 1. $Bb = \sqrt{x'^2 + y'^2}^{\frac{1}{2}}$
 Now because EB is perpendicular to a Tangent at the Point B, and Bb is supposed to coincide with that Tangent, the Triangles Bnb and BCH are Similar (For $\angle EBb = \angle CBn$, and therefore, the $\angle EBn$ being common, the Angles nBb and CBH are equal $\therefore \angle Bbn = \angle BCH$, *ergo*, &c.) and the $\angle Bbm$ being right, the Triangles mnb and $b n B$ are similar also, as are likewise therefore the Triangles $b n m$ and BCH . Wherefore $Bn : nb :: BC : CH$, *i. e.* $x' : y' :: y : \frac{yy'}{x'} = CH$:

And therefore (by 47 E. 1.) $BH = \sqrt{BC^2 + CH^2}^{\frac{1}{2}} = \sqrt{y^2 + \frac{y^2 y'^2}{x'^2}}^{\frac{1}{2}} = \frac{y}{x'} \times \sqrt{x'^2 + y'^2}^{\frac{1}{2}}$

and $AH = x + \frac{yy'}{x'}$. Again $Bn : nb :: bn : nm$

M 2

i. e.

$$i. e. x' : y' :: y' : \frac{y'^2}{x'} = nm \therefore Bm = x' + \frac{y'^2}{x'}$$

Now because the Direction of the Curve ABG approaches continually nearer to a Parallelism with the Axis AX; if we suppose the Absciss (x) to flow with an equable or uniform Motion; that is, supposing x' or $\dot{x}$ to be invariable; it is plain that the Increment of the Ordinate (y) or the Velocity, with which it flows, must continually decrease; and therefore the Increment or Fluxion of this Increment, or the second Increment of y , will be Negative. Therefore Hb , the Increment of

$$AH, viz. the Increment of $x + \frac{yy'}{x'}$ will be $x' + \frac{y'^2 - yy''}{x'}$. Now the Triangles EBm and$$

and EHm are similar, as is evident; therefore $Bm - Hb : Bm :: (BE - HE =) BH :$

$$BE, i. e. \frac{yy''}{x'} : x' + \frac{y'^2}{x'} :: \frac{y}{x'} \times \sqrt{x'^2 + y'^2}^{\frac{1}{2}} :$$

$$\frac{\sqrt{x'^2 + y'^2}^{\frac{3}{2}}}{x'y''} = BE ; \text{ or, substituting the Fluxion}$$

$$\text{for the Increment (Art. 3.) } BE = \frac{\sqrt{\dot{x}^2 + \dot{y}^2}^{\frac{3}{2}}}{\dot{x}\ddot{y}}.$$

63. Note, If x flow with an uniform and y with an accelerated Motion, *i. e.* if $\dot{x}$ be invariable,

riable, as above, and the Fluxion of y affirmative (as it will be when the Curve is Convex towards its Axis) the general Expression for BE, the Radius of Curvature, will be $\frac{\dot{x}^2 + \dot{y}^2|^{\frac{3}{2}}}{-\dot{x}\ddot{y}}$.

64. Hence, because we may substitute 1 for any invariable Fluxion, if we put $\dot{x}=1$, the general Expression for BE the Radius of Curvature will be $= \frac{1 + \dot{y}^2|^{\frac{3}{2}}}{\ddot{y}}$ when the Fluxion of y is Negative, or the Curve is Concave towards its Axis; and $= \frac{1 + \dot{y}^2|^{\frac{3}{2}}}{-\ddot{y}}$ when the Fluxion of y is Affirmative, or the Curve is Convex towards its Axis. Wherefore, if we put the Equation of the given Curve, which expresses the Relation between the Absciss and Ordinate into Fluxions, making $\dot{x}=1$; and put this fluxional Equation into Fluxions again, still substituting 1 for $\dot{x}$, and making the Fluxion of y Negative when the Curve is Concave, and Affirmative when Convex towards its Axis, the Values of the *Second* and *Square* of the *First* Fluxion of y may be had or determined; which therefore being substituted for them in one of these general Expressions, *viz.* in the first, when the Fluxion of y is Negative, and in the

the second, when Affirmative, will give a definitive Expression for B E, or the Radius of Curvature required.

65. *N. B.* The shortest Radius, or vertical Distance A D may be obtained by substituting the Values of x and y in the general Expression for the subnormal CH, which was found (*Art.*

62.) $= \frac{yy'}{x'}$, or (substituting x for x' and y for y')

$\frac{yy}{x}$; and then making x and y vanish in the definitive Expression which will be then found: For the Expression for CH being the same at whatever Point of the Curve B is taken, therefore if it be taken at A where x and y vanish or become $=0$, the Point C must of consequence coincide with the Point A, and the Points E and H with D, *ergo*, &c.

66. *N. B.* The substituting Unity, or 1, rather than any other Number, for an invariable Fluxion, has no manner of Effect on the Working, otherwise than by making the Operation much less laborious; and, in reality, it is no more than making Unity the Standard of the other Fluxions, or reducing the other Fluxions to a Comparison with 1.

E x-

EXAMPLE I.

67. To find a definitive Expression for the Radius of Curvature BE (see the last Fig.) as also AD the Distance of the Vertices of the Evolute and Involute Curves; supposing the involute Curve ABG to be the common or apollonian Parabola.

THE Equation of the common Parabola is $ax=y^2$ (see Art. 38.) which in Fluxions is $ax=2y\dot{y}$; or, if we write 1 for $\dot{x}$, it is $a=2y\ddot{y}$;

therefore $\dot{y}=\frac{a}{2y}=\frac{a}{2 \times ax}^{\frac{1}{2}}$, for y is $=ax^{\frac{1}{2}}$ by

the Equation of the Curve: Now the Fluxion of this Equation again (the Curve being Concave towards its Axis, and therefore the Fluxion of $\dot{y}$ Negative) is $-\ddot{y}=\frac{-a^2}{4 \times ax}^{\frac{1}{3}}$ whence $\dot{y}^3$

$=\frac{a^2}{4 \times ax}=\frac{a}{4x}$ and $\ddot{y}=\frac{a^2}{4 \times ax}^{\frac{1}{2}}$. Now, if for $\dot{y}^3$ and $\ddot{y}$, we substituted these their Values, we

shall have $\frac{1+\dot{y}^2}{\ddot{y}}$, the general Expression for the Radius of Curvature BE (Art. 64.) =

1 +

$$\frac{1 + \frac{a^2}{4ax} \times 4 \times ax}{a^2} = \frac{4ax + a^2}{2a^2},$$
 which is the definitive Expression required. And by substituting $\frac{a}{2y}$ for y in ($\frac{yy}{x}$ Art. 65. or x being 1.) yy we have $\frac{a}{2}$ or $\frac{1}{2}a = AD$ the vertical Distance; which same truth may be infer'd from the Expression for BE, for when the Radius becomes the vertical Distance, that is, when the Point B coincides with A, x vanishes, and therefore striking $4ax$ out of the said Expression, we have $\frac{a^2}{2a^2} = \frac{1}{2}a = AD$ as before.

EXAMPLE II.

68. Let $y^m = x$ express the Nature of all Parabolas universally (see Art. 39.) to find the Radius of Evolution or Curvature, and vertical Distance; as also the Consequences on these three Suppositions respectively, viz. $m=2$, m more than 2, m less than 2.

THE Fluxion of this Equation, making $x=1$, is $my^{m-1}\dot{y}=1$; and the Fluxion of this again (supposing y Negative) is $m-1 \times my^{m-2}\dot{y}^2$
— my

$-my^{m-1}\ddot{y}=0$, therefore $\dot{y}=\frac{1}{my^{m-1}}$, and $\dot{y}^2=$

$\frac{1}{m^2y^{2m-2}}$, and $\dot{y}=\frac{m-1 \times my^{m-2}}{my^{m-1}}\dot{y}^2=$ (by di-

viding both Numerator and Denominator by

my^{m-2}) $\frac{m-1}{y}\dot{y}^2=$ (by writing for $\dot{y}^2$ its equal)

$\frac{m-1}{m^2y^{2m-1}}$. Now by substituting for $\dot{y}^2$ and $\dot{y}$

these their Values, we shall have the general

Expression for the Radius of Curvature $\frac{1+\dot{y}^2}{\dot{y}}$

(Art. 64.) $=\frac{1+\frac{1}{m^2y^{2m-2}}}{\frac{1}{my^{m-1}}} \times m^2y^{2m-2} =$

$\frac{m^2y^{2m-2}+1}{m-1} \times \frac{1}{m}y^{2-m}$ = the Radius of Curvature re-

quired. And by substituting $\frac{1}{my^{m-1}}$ for $\dot{y}$ in

$\left(\frac{y\dot{y}}{\dot{x}}, \text{ or, } \dot{x} \text{ being } =1,\right) y\dot{y}$ (see Art. 65.) we

have the Subnormal $=\frac{y}{my^{m-1}}=\frac{y^2}{my^m}=\frac{y^{2-m}}{m}$,

i. e. (if $m=2$,) $=\frac{y^0}{2}=\frac{1}{2}$; if m be more than

2, then $2-m$ will be a negative Index, and

N

con-

EXAMPLE III.

PUT

PUT OF or OK = a , AC = x , CB = FN
 y , Sine NG = s , Arch GF = z . Now by
 the Nature of the Curve (*see Art. 45.*) the
 Arch KG = GB and therefore the Arch
 FG = GN + AC, or AC = Arch FG - GN,
i. e. $x = z - s$, or, (because by the Property of
 Circles $NG = \sqrt{KN \times NF}$), *i. e.* $s = \sqrt{2ay - y^2}$,
 $x = z - \sqrt{2ay - y^2}$ which in Fluxions, making

$$\dot{x} = 1, \text{ is } 1 = \dot{z} + \frac{yy - ay}{2ay - y^2}^{\frac{1}{2}}, \text{ but } \dot{z} = \sqrt{\dot{s}^2 + \dot{y}^2}^{\frac{1}{2}} =$$

$$\frac{\sqrt{ay - yy}^2}{2ay - y^2} + \dot{y}^2^{\frac{1}{2}} = \frac{ay}{2ay - y^2}^{\frac{1}{2}}, \text{ therefore } 1 =$$

$$\frac{ay}{2ay - y^2}^{\frac{1}{2}} + \frac{yy - ay}{2ay - y^2}^{\frac{1}{2}}, \text{ i. e. } 1 = \frac{yy}{2ay - y^2}^{\frac{1}{2}} \therefore \dot{y} =$$

$$\frac{2ay - y^2}{y} \text{ and this in Fluxions again, because}$$

the Fluxion of y is Negative, is $-\dot{y} =$

$$\frac{ay\dot{y} - y^2\dot{y}}{2ay - y^2}^{\frac{1}{2}} - \dot{y} \times \frac{2ay - y^2}{y^2}^{\frac{1}{2}} = \frac{-ay}{y \times 2ay - y^2}^{\frac{1}{2}} =$$

(by substituting for y , its equal,) $-\frac{a}{y^2}$, or $\dot{y} =$

$\frac{a}{y^2}$. Now by substituting $\frac{2ay - y^2}{y^2}$ for y^2 and

$\frac{a}{y^2}$ for $\dot{y}$ we shall have, by *Art. 46.* $\frac{1 + \dot{y}^2}{\dot{y}} =$

1 +

$$\frac{1 + \frac{2ay - y^2}{y^2} \times y^2}{a} = \frac{2ay}{ay} = BE \text{ the Radius}$$

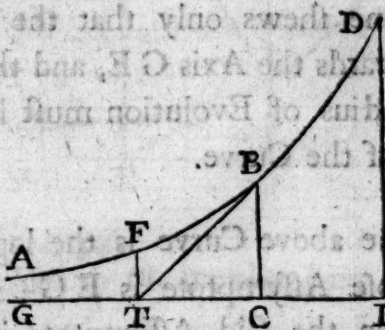
of Curvature required : Whence, by Analogy $BE : 2ay^{\frac{1}{2}} :: 2ay : ay$, that is, $BE : 2ay^{\frac{1}{2}} :: 2 : 1$, but $2ay^{\frac{1}{2}} = FG$ (for by 47 E. 1. $\overline{GN}^2 + \overline{NF}^2 = \overline{FG}^2$, and by the Property of Circles $\overline{GN}^2 = KN \times NF$, therefore $\overline{FG}^2 = KN \times NF + \overline{NF}^2 = 2a - y \times y + y^2 = 2ay$ and $FG = 2ay^{\frac{1}{2}}$.) therefore $BE = 2GF$: And as a Tangent at the Point B is parallel to the Chord KG, by Art. 45. and the Angle KGF is right, therefore, by Art. 61. Cor. 2. BE will be parallel to the Chord GF. — By Art. 65. by substituting $\frac{2ay - y^2}{y}$ for its equal y in $(\frac{yy}{x}$, or, x being = 1) yy , we have $CH = \frac{2ay - y^2}{y}$ and by making y vanish it becomes = 0, so that the Vertices of the evolute and involute Curves coincide.

EXAMPLE IV.

70. To find the Radius of Curvature at any Point B of the Curve AD, whose Nature is such, that the Triangle CBT, made of the Ordinate,

DOCTRINE of FLUXIONS. 93

ordinate, Tangent, and Subtangent, is always proportional to the Ordinate CB, or, whose Subtangent CT is always the same.



Put the given Subtangent $CT = a$, $GC = x$, $CB = y$; then by *Art.* 35. $\frac{\dot{x}y}{y} = a$, *i. e.* (if $\dot{x}$ be made $= 1$,) $\frac{y}{\dot{y}} = a \therefore \dot{y} = \frac{y}{a}$, therefore $\dot{y}^2 = \frac{y^2}{a^2}$ and, (because here y flows with an accelerated Motion, or, its second Fluxion is Affirmative,) $\ddot{y} = \frac{\dot{y}}{a}$, *i. e.* (by substituting $\frac{y}{a}$ for $\dot{y}$ its equal,) $\ddot{y} = \frac{y}{a^2}$: Now by substituting for $\dot{y}^2$ and $\ddot{y}$ these their Values in $\frac{1 + \dot{y}^2}{-\ddot{y}}$ (*Art.* 64.) we

64.) we shall have
$$\frac{\sqrt{1 + \frac{y^2}{a^2}} \times a^2}{-y} = \frac{\sqrt{a^2 + y^2}}{-ay} =$$

the Radius of Curvature sought: where the negative Sign shews only that the Curve is Convex towards the Axis GE, and that therefore the Radius of Evolution must lie on the other Side of the Curve.

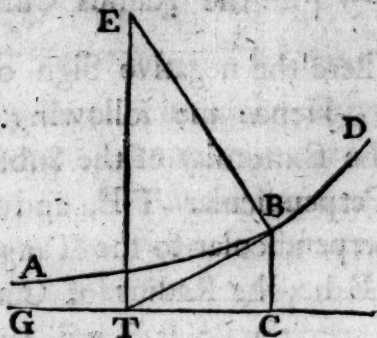
Note, The above Curve is the logarithmic Curve, whose Asymptote is EG; so called, because when the said Asymptote is divided into any Number of equal Parts, as in the Points G, T, C, E, the Ordinates to these Points will be in geometrical Progression, i. e. GT, GC, &c. will be the Logarithms of the Ordinates TF, CB, &c.

EXAMPLE V.

71. To find the Radius of Curvature for any Point B of the Curve AD, whose Nature is such, that the Tangent BT is every where equal to the same given Line = a.

Put GC = x, CB = y, then, by 47 E. 1,

$$\sqrt{TB^2 - BC^2} = CT, \text{ i. e. } \sqrt{a^2 - y^2} = CT =$$
(by



(by *Art.* 35.) $\frac{\dot{x}y}{\dot{y}}$ or, making $\dot{x}=1$, $\frac{y}{a^2-y^2} =$

$$= \frac{y}{\dot{y}} \therefore \dot{y} = \frac{y}{a^2-y^2}, \text{ and } \dot{y}^2 = \frac{y^2}{a^2-y^2}, \text{ and}$$

supposing the Fluxion of y affirmative, $\ddot{y} =$

$$\frac{\dot{y} \times a^2 - y^2 + \frac{y^2 \dot{y}}{a^2 - y^2}}{a^2 - y^2}, \text{ i. e. by substituting}$$

$$\text{for } \dot{y} \text{ its equal, } \ddot{y} = \frac{y + \frac{y^3}{a^2 - y^2}}{a^2 - y^2} = \frac{a^2 y}{a^2 - y^2}.$$

Now by writing for $\dot{y}^2$ and $\dot{y}$ these their Values

in $\frac{1 + \dot{y}^2}{-\dot{y}}$ (*see Art.* 64.) we shall have

$$\frac{1 + \frac{y^2}{a^2 - y^2}}{-a^2 y} \times a^2 - y^2, \text{ i. e. } \frac{a^2 \frac{1}{2} \times a^2 - y^2 \frac{1}{2}}{-a^2 y} \text{ or } -a$$

72. LET ABG be the Curve; A the central Point, or that from which all the Ordinates issue; and BE, the Radius of Curvature at the Point B, *i. e.* let E be supposed in the evolute Curve: Conceive Ab and Eb indefinitely near to AB and EB; and AD, Ad, perpendicular to EB, Eb; then will the Points D and m be indefinitely near to a Coincidence, and Bm and Am may be taken as equal to BD and AD, the Difference being indefinitely small. Now if with the Radius AB, the little circular Arch Bn be described, (which may be considered as a little right Line, as may also the Curve Bb,) the little right angled Triangle bnb will be similar to the right angled Triangle BDA: (For $\angle ABn = \angle EBb$ therefore, $\angle EBn$ being common, $\angle ABD = \angle nBb$, and therefore, the Angles at D and n being right, $\angle BAD = \angle Bbn$;) therefore $bB : Bn :: AB : BD$, that is, (if we put $AB = y$, $Bn = x'$, $nb = y'$) because by 47 E. 1.

$$\sqrt{x'^2 + y'^2} = Bb, \sqrt{x'^2 + y'^2} : x' :: y : \frac{x'y}{\sqrt{x'^2 + y'^2}}$$

= BD, or Bm; or, substituting x for x' and

$$y \text{ for } y', \frac{xy}{\sqrt{x^2 + y^2}} = BD \text{ or } Bm. \text{ Again } Bb$$

$$: bn :: BA : AD, \text{ that is, } \sqrt{x'^2 + y'^2} : y' :: y :$$

O

yy'

$\frac{yy'}{\dot{x}^2 + \dot{y}^2} = AD$ or Am , or $\frac{y\ddot{y}}{\dot{x}^2 + \dot{y}^2} = AD$
or Am , whose Fluxion, (supposing $\dot{x}$ invariable,) is

$$\frac{\dot{y}^2 + y\ddot{y} \times \dot{x}^2 + \dot{y}^2}{\dot{x}^2 + \dot{y}^2} - \frac{y\ddot{y} \times y\ddot{y}}{\dot{x}^2 + \dot{y}^2} = \frac{\dot{x}^2 \dot{y}^2 + \dot{y}^4 + y\dot{x}^2 \ddot{y}}{\dot{x}^2 + \dot{y}^2}$$

$= md$. But by the similar Triangles EBb and Emd , we have $Bb - md : Bb :: BE - Em$, or, $mB : BE$; or, $\frac{\dot{x}^2 + \dot{y}^2}{\dot{x}^2 + \dot{y}^2} -$

$$\frac{\dot{x}^2 \dot{y}^2 - \dot{y}^4 - y\dot{x}^2 \ddot{y}}{\dot{x}^2 + \dot{y}^2} = \frac{\dot{x}^4 + \dot{x}^2 \dot{y}^2 - y\dot{x}^2 \ddot{y}}{\dot{x}^2 + \dot{y}^2} : \dot{x}^2 + \dot{y}^2$$

$$:: \frac{\dot{x}y}{\dot{x}^2 + \dot{y}^2} : \frac{y \times \dot{x}^2 \times \dot{y}^2}{\dot{x}^3 + \dot{x}\dot{y}^2 - y\dot{x}\ddot{y}} = BE, \text{ which is}$$

a general Expression for the Radius of Evolution, or Curvature, of all Curves referred to a fixt or central Point, when $\dot{x}$ is invariable. Wherefore,

73. If $\dot{x}$ be made $= 1$, the general Expression for BE , the Radius of Curvature, will be

$$= \frac{y \times 1 + \dot{y}^2}{1 + \dot{y}^2 - y\ddot{y}}, \text{ and if we put the Equation of}$$

the given Spiral into Fluxions, making $\dot{x} = 1$, and put this resulting Equation into Fluxions again, the Values of the second and Square of the first Fluxions of y being determined, and substituted for them in this general Expression, will

give

$\therefore \frac{ay}{b} = \frac{bx}{y}$ or, making $\dot{x} = 1$, $\frac{ay}{b} = \frac{b}{y}$ which

Equation reduced gives $\dot{y} = \frac{b^2}{ay}$: Now the

Fluxion of this again is $\ddot{y} = \frac{-ab^2\dot{y}}{a^2y^2} =$ (by

writing $\frac{b^2}{ay}$ for $\dot{y}$) $\frac{-b^4}{a^2y^3}$, and $\dot{y}^2 = \frac{b^4}{a^2y^2}$. And

if we substitute for $\dot{y}^2$ and $\ddot{y}$ these their Values

we shall have $\frac{y \times \sqrt{1 + \dot{y}^2}^{\frac{3}{2}}}{1 + \dot{y}^2 - y\ddot{y}}$ (Art. 73.) =

$$\frac{y \times \sqrt{1 + \frac{b^4}{a^2y^2}}^{\frac{3}{2}}}{1 + \frac{b^4}{a^2y^2} + \frac{b^4}{a^2y^2}} = \frac{\sqrt{a^2y^2 + b^4}^{\frac{3}{2}}}{a^2y^2 + 2ab^4} = \text{BE the Ra-}$$

dius of Curvature sought.

EXAMPLE II.

75. Let the Nature of any Spiral be expressed by this Equation $ay^m = b^m x$, where m stands for any whole Number or Fraction ad libitum: To find the Radius of Curvature for any Point.

THIS Equation in Fluxions is $amy^{m-1}\dot{y} = bmx$, i. e. $\frac{amy^{m-1}}{b^m}\dot{y} = \dot{x} = \frac{bx}{y}$ by the last Ex-

ample,

ample, *i. e.* (by making $\dot{x}=1$) $\frac{amy^{m-1}}{b^m} \dot{y} = \frac{b}{y}$

or $\dot{y} = \frac{b^{m+1}}{amy^m}$. Now, this put into Fluxions

again is, $\ddot{y} = \frac{-am^2y^{m-1} \times \dot{y}b^{m+1}}{a^2m^2y^{2m}} = \frac{-b^{m+1}\dot{y}}{ay^{m+1}} =$

(by substituting for $\dot{y}$, its equal) $\frac{-b^{2m+2}}{a^2my^{2m+1}}$; and

$\dot{y}^2 = \frac{b^{2m+2}}{a^2m^2y^{2m}}$ and if we write for $\dot{y}^2$ and $\ddot{y}$,

these their Values, in $\frac{y \times 1 + \dot{y}^2}{1 + \dot{y}^2 - y\ddot{y}}$ (*Art.* 73.)

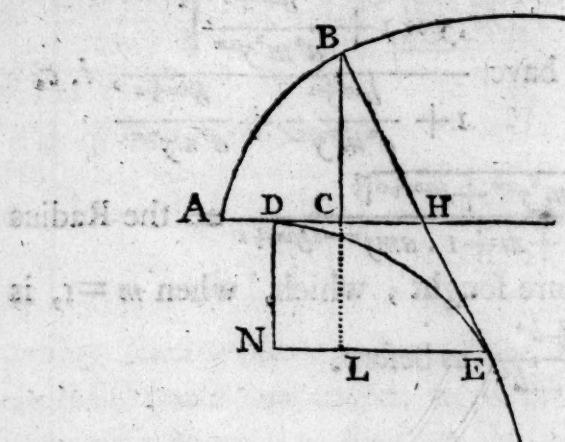
we shall have $\frac{y \times 1 + \sqrt{\frac{b^{2m+2}}{a^2m^2y^{2m}}}}{1 + \frac{b^{2m+2}}{a^2m^2y^{2m}} + \frac{b^{2m+2}}{a^2my^{2m}}}$, *i. e.*

$\frac{\sqrt{a^2m^2y^{2m} + b^{2m+2}}}{a^3m^3y^{3m-1} + m+1 \cdot amy^{m-1}b^{2m+2}} =$ the Radius
of Curvature sought; which, when $m=1$, is
 $= \frac{a^2y^2 + b^4}{a^3y^2 + 2ab^4}$ as before.

C H A P. VII.

Of finding the Nature of the Evolutes of given Involute CURVES.

AS it is absolutely necessary for the Learner to be well acquainted with the foregoing Chapter, before he enters upon this, we shall not here define the Meaning of Evolute and Involute Curves, it being sufficiently explained in that.



76. LET BE, be the Radius of Evolution at any Point B of the involute Curve AB, whose Absciss is $AC = x$, and Ordinate $CB = y$, parallel to HA draw EN; produce BC to L, and equal and parallel to CL draw DN; then

then will the Triangles B H C and B E L be similar; and therefore $BH : HC :: BE : EL$,

$$i. e. (Art. 62.) \frac{y}{\dot{x}} \times \overline{\dot{x}^2 + \dot{y}^2}^{\frac{1}{2}} : \frac{y\ddot{y}}{\dot{x}} :: \frac{\overline{\dot{x}^2 + \dot{y}^2}^{\frac{3}{2}}}{\ddot{y}} :$$

$$\frac{\dot{y} \times \overline{\dot{x}^2 + \dot{y}^2}}{\ddot{y}} = EL, \text{ and } CH : CB :: EL :$$

$$LB, i. e. \frac{y\ddot{y}}{\dot{x}} : y :: \frac{\dot{y} \times \overline{\dot{x}^2 + \dot{y}^2}}{\ddot{y}} : \frac{\dot{x} \times \overline{\dot{x}^2 + \dot{y}^2}}{\ddot{y}} =$$

LB, and these are generally Expressions for EL, and LB, when $\dot{x}$ is considered as invariable, and the Fluxion of y as Negative. Hence

77. If $\dot{x}=1$, and the Fluxion of y be negative, the general Expression for EL will be =

$$\frac{\dot{y} \times \overline{1 + \dot{y}^2}}{\ddot{y}}, \text{ and that for LB} = \frac{1 + \dot{y}^2}{\ddot{y}}. \text{ Now by}$$

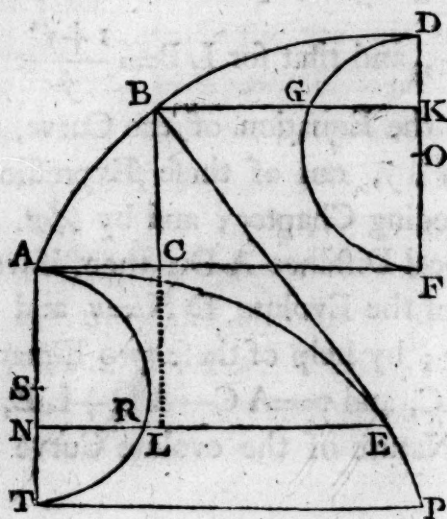
help of the Equation of the Curve, exterminate $\dot{y}$, $\dot{y}^2$, $\ddot{y}$, out of these Expressions, as in the preceding Chapter; and by *Art. 65.* find the vertical Distance AD; then, if we put the Absciss of the Evolute $DN=u$ and Ordinate $NE=v$; by help of these two Equations $u=BL-BC$, and $v=AC-AD+LE$, we may get the Nature of the evolute Curve DE required.

78. *Note*, If the given Involute be Convextowards its Axis and the Fluxions of x and y increase together, the 2^d Fluxion of y will be affirmative, and the general Expressions for EL and LB will

will be $\frac{\dot{y} \times 1 + \dot{y}^2}{-\dot{y}}$ and $\frac{1 + \dot{y}^2}{-\dot{y}}$; where the Negative Sign only shews that the Points E and L must be taken on the concave side of the Curve, that is, on the other side of the Curve with regard to x and y .

EXAMPLE I.

79. To find the Nature of that Curve by whose Evolution the Cycloid ABD is described.



Put $AC = x$, $CB = y$, Arch $FG = z$, OD
 or $OF = a$; then (Art. 69.) $\dot{y} = \frac{2ay - y^2}{y}$, $\dot{y}^2 =$
 $2ay - y^2$

$\frac{2ay-y^2}{y^2}$, and $\ddot{y} = \frac{a}{y^2}$ wherefore (see Art.

$$77.) \quad B L = \frac{1+\dot{y}^2}{\ddot{y}} = \frac{1+\frac{2ay-y^2}{y^2} \times y^2}{a} =$$

$$2y, \text{ and } LE = \frac{\dot{y} \times \frac{1+\dot{y}^2}{\ddot{y}}}{\ddot{y}} = \dot{y} \times 2y = 2 \times \sqrt{2ay-y^2}^{\frac{1}{2}}$$

Hence, if we put the Absciss $AN = u$, and Ordinate $NE = v$, we have $u = 2y - y = y$, and $v = x + 2 \times \sqrt{2ay-y^2}^{\frac{1}{2}}$, i. e. (because $x = z - \sqrt{2ay-y^2}^{\frac{1}{2}}$, Art. 69.) $v = z + \sqrt{2ay-y^2}^{\frac{1}{2}}$, or, writing u for y its equal, $v = z + \sqrt{2au-u^2}^{\frac{1}{2}}$. Wherefore the evolute Curve $A E Q$ is a Cycloid, and equal to the given Cycloid $A B D$: For let $AS = ST = a$, then (AN being $= FK$), $AR = FG = z$, and $NR = \sqrt{2au-u^2}^{\frac{1}{2}} = KG$, and therefore $AR + RN = z + \sqrt{2au-u^2}^{\frac{1}{2}}$, i. e. $AR + RN = NE$, which is the Property of the Cycloid, therefore AEP is a Cycloid; and because $AT = FD$, therefore the Cycloids AEP and ABD are equal.

EXAMPLE II.

80. To find the Nature of the Evolute of the Curve AD , whose Tangent BT , is every where equal to the same given Line $= a$.

P

LET

$$\frac{1+\dot{y}^2}{\ddot{y}} = \frac{1+\frac{y^2}{a^2-y^2}}{\frac{a^2 y}{a^2-y^2}} = - : \frac{a^2-y^2}{y}, \text{ and LE} =$$

$$\dot{y} \times \frac{1+\dot{y}^2}{\ddot{y}} = \dot{y} \times \text{BL} = \frac{y}{a^2-y^2} \times - : \frac{a^2-y^2}{y}$$

$= -\frac{a^2-y^2}{y^2}$, where the negative Sign shews that the Points L and E must be taken on the concave Side of the Curve. Hence we have

$$u = \frac{a^2-y^2}{y} + y - a \text{ which gives } y = \frac{a^2}{u+a} \text{ and } \dot{y}$$

$$= -\frac{a^2 \dot{u}}{u+a^2} : \text{ also } v = x + \sqrt{a^2-y^2}, \text{ and } \dot{v} =$$

$$\dot{x} - \frac{y \dot{y}}{a^2-y^2}, \text{ i. e. (because (Art. 36.) } -\frac{\dot{x} y}{\dot{y}}$$

$$= \sqrt{a^2-y^2} \text{ or } \dot{x} = -\frac{\dot{y}}{y} \times \sqrt{a^2-y^2}) \dot{v} = -\frac{\dot{y}}{y} \times$$

$$\sqrt{a^2-y^2} - \frac{y \dot{y}}{a^2-y^2} = \frac{-a^2 \dot{y}}{y \times a^2-y^2}; \text{ or, sub-}$$

stituting for y and $\dot{y}$, their above Values, $\dot{v} =$

$$\frac{a \dot{u}}{u^2 + 2au}, \text{ which is an Equation for the evo-$$

lute Curve AD, and is an Equation for the catenary Curve; therefore the Evolute AD is the Catenary*.

Note, The above Equation of the Evolute

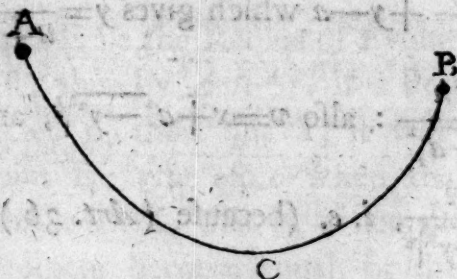
may be found otherwise, thus; Let $En = u'$,
 $ne = v'$; then, the Triangles enE and EBT
 being Similar, as $en : nE :: TB : BE$, i. e.

$$v' : u' :: a : \frac{au'}{v'} = BE \text{ or } \frac{a\dot{u}}{v} = BE : \text{ But } BE$$

$$= \sqrt{ET^2 - TB^2}^{\frac{1}{2}}, \text{ i. e. (because } ET = XN$$

$$= u + a) BE = \sqrt{u^2 + a^2 - a^2}^{\frac{1}{2}} = \sqrt{u^2 + 2au}^{\frac{1}{2}} \text{ there-}$$

$$\text{fore } \frac{a\dot{u}}{v} = \sqrt{u^2 + 2au}^{\frac{1}{2}} \text{ and } \dot{v} = \frac{a\dot{u}}{u^2 + 2au}^{\frac{1}{2}}.$$



* The Catenary is a Curve as ACB formed
 by a flexible Line hanging freely from two
 Points of Suspension A, B, whether these
 Points be horizontal or not.

81. To find the Evolute of a Spiral, you
 must find the Radius of Curvature at several
 Points, which will give as many Points in the
 evolute Curve; then a curve Line drawn thro'
 those Points so found, will be the Evolute
 sought.

C H A P. VIII.

Of infinite SERIES.

IT being absolutely necessary for the Learner to understand something of infinite Series, before he can find the Fluents, or flowing Quantities, of Fluxions express'd after a fractional Manner, or wherein there are Surds or radical Quantities; and because the next Chapter treats of the finding the flowing Quantities or Fluents of fluxional Expressions; we thought it not improper to add this Chapter, tho' it is, in some Measure, foreign to our Business.

P R O B. I.

To reduce fractional Quantities into infinite Series.

E X A M P L E I.

82. *Let it be required to reduce $\frac{b}{a+x}$ into an infinite Series.*

Place $a+x$ as a Divisor and b as a Dividend
and

and divide as in common Division, till you have 4, 5, 6, or more Terms in the Quotient ; after which (in most Cases) you may find as many Terms as you please, by considering the Law of the Progression of the Terms already found. Thus the four first Terms being $\frac{b}{a} -$

$\frac{bx}{a^2} + \frac{bx^2}{a^3} - \frac{bx^3}{a^4}$, the Law of the Continuation of the Division, or Series, is plain ; for the Quotient consists of an infinite Series of Terms whose Numerators are the Powers of x less by 1 than the Number of the Order, multiplied by b ; and Denominators the Power of a , whose Indices are the Number of the Order of the Terms : And having its Signs changed alternately, thus the fifth Term will be $+ \frac{bx^4}{a^5}$

and the sixth Term $- \frac{bx^5}{a^6}$; and if an infinite

Number or Series of Terms be so taken, it will be the exact Quotient of the Division, and consequently exactly equal to the given fractional Expression ; but (generally) a few of the first Terms of the Series are near enough the Truth for any Purpose.

Or, if we put x before a in the Denominator of the above given fractional Expression,

i. e.

i. e. if the Divisor be express'd thus $x+a$ instead of $a+x$; the Quotient or Series will be

$$\frac{b}{x} - \frac{ba}{x^2} + \frac{ba^2}{x^3} - \frac{ba^3}{x^4}, \text{ \&c. whence the Law}$$

of the Continuation of the Series may be observed as before.

83. *Note.* There will always be as many Quotients or infinite Series, as there are Terms in the Denominator or Divisor, though only one true; and to find this, you must always place the greatest Terms in the Divisor and Dividend first, *i. e.* if, in the last Example, for Instance, a be greater than x , then a must be

the first Term in the Divisor, and $\frac{b}{a} - \frac{bx}{a^2} + \frac{bx^2}{a^3} - \frac{bx^3}{a^4}, \text{ \&c. will be the true Series ; but if}$

x be greater than a , then x must be the first

Term in the Divisor, and $\frac{b}{x} - \frac{ba}{x^2} + \frac{ba^2}{x^3} - \frac{ba^3}{x^4},$

\&c. will be the true Series ; the other, then being a diverging one, and consequently the further you go in the Series, the farther it will be from the Truth.

EXAMPLE II.

84. Let it be required to throw $\frac{a^3}{a^2+2ax+x^2}$ into an infinite Series, supposing a to be greater than x .

OPERATION.

$$\begin{array}{r}
 a^3+2ax+x^2 \quad a^2 \dots\dots\dots (1-\frac{2x}{a}+\frac{3x^2}{a^2}-\frac{4x^3}{a^3} \\
 \hline
 \frac{a^3+2ax+x^2}{0-2ax-x^2} \\
 \hline
 -2ax-4x^2-\frac{2x^3}{a} \\
 \hline
 0+3x^2+\frac{2x^3}{a} \\
 \hline
 3x^2+\frac{6x^3}{a}+\frac{3x^4}{a^2} \\
 \hline
 0-\frac{4x^3}{a}-\frac{3x^4}{a^2} \\
 \hline
 \frac{4x^3}{a}-\frac{8x^4}{a^2}+\frac{4x^5}{a^3} \\
 \hline
 0+\frac{5x^4}{a^2}+\frac{4x^5}{a^3}
 \end{array}$$

Now from these four Terms, it is easy to see that the Law of Continuation is such, that the

the Numerators are the Powers of x , whose Exponents are 1 less than the Number of the Order, multiplied into the said Number; and the Denominators, the Powers of a , whose Exponents are the same with those of the correspondent Numerators; the Signs being changed alternately: So that the fifth Term is $+\frac{5x^4}{a^4}$, the sixth Term $-\frac{6x^5}{a^5}$; and so on.

P R O B. II.

To reduce a compound surd Quantity into an infinite Series, *i. e.* to free a compound Expression from Surds, by throwing it into an infinite Series.

E X A M P L E I.

85. Let it be required to throw $\sqrt{a^2+x^2}$ or $(a^2+x^2)^{\frac{1}{2}}$ into an infinite Series.

Note. In Order to have a true Series, the greatest Term must be always plac'd first, as in *Prob. I.* Therefore, supposing a^2 to be greater than x^2 ,

Take the Square Root of a^2 , which is a , for the first Term in the Root (see the Operation below,) then this squar'd and subtracted

Q

from

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from $a^2 + x^2$ leaves $+x^2$. And this Remainder divided (as in the common Extraction of the Square Root) by $2a$, viz. the double of the first Term, gives $\frac{x^2}{2a}$ for the second Term in the Root, which, together with the double of the first Term, being multiplied by $\frac{x^2}{2a}$ the said second Term, gives $x^2 + \frac{x^4}{4a^2}$ and this subtracted from x^2 leaves $-\frac{x^4}{4a^2}$ which divided by the double of the two 1st Terms in the Root, viz. by $2a + \frac{x^2}{a}$ gives $-\frac{x^4}{8a^3}$ for the 3^d Term in the Root which, together with the double of the two first Terms, viz. $2a + \frac{x^2}{a}$, multiplied by $-\frac{x^4}{8a^3}$ the said third Term, gives $-\frac{x^4}{4a^2} - \frac{x^6}{8a^4} + \frac{x^8}{64a^6}$ and this subtracted from $-\frac{x^4}{4a^2}$ leaves $\frac{x^6}{8a^4} - \frac{x^8}{64a^6}$, which divided by the double of the three Terms of the Root already found, viz. by $2a + \frac{x^2}{a} - \frac{x^4}{4a^3}$ gives $\frac{x^6}{16a^5}$ for the fourth

Term

Term in the Root, and after the same Manner may be found any Number of Terms in the Root; and after the Law of the Progreſſion or Continuation is diſcovered, the Series may be continued on at Pleaſure.

O P E R A T I O N.

$$\begin{array}{r}
 a^3 + x^3 \left(a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} \right. \\
 2a) \dots \frac{a^2}{0 + x^2} \\
 \quad + x^2 + \frac{x^4}{4a^2} \\
 \hline
 2a + \frac{x^2}{a}) \dots \frac{x^4}{4a^2} \\
 \quad - \frac{x^4}{4a^2} - \frac{x^6}{8a^4} + \frac{x^8}{64a^6} \\
 \hline
 2a + \frac{x^2}{a} - \frac{x^4}{4a^3}) \dots \frac{x^6}{8a^4} - \frac{x^8}{64a^6}
 \end{array}$$

E X A M P L E II.

86. Let it be required to throw $\sqrt{x - x^2}^{\frac{1}{2}}$ into an infinite Series; where x is ſuppoſed to be leſs than Unity, or x greater than x^2 .

O P E R A -

OPERATION.

The divisions are placed wrong.

$$\begin{array}{r}
 x - x^2 \left(x^{\frac{1}{2}} - \frac{1}{2}x^{\frac{3}{2}} - \frac{1}{8}x^{\frac{5}{2}} - \frac{1}{16}x^{\frac{7}{2}} - \frac{1}{128}x^{\frac{9}{2}} \right) \\
 \hline
 -x^2 + \frac{1}{2}x^3 \\
 \hline
 2x^{\frac{3}{2}} - x^{\frac{5}{2}} \dots \dots \dots \frac{1}{4}x^3 \\
 \hline
 -\frac{1}{2}x^3 + \frac{1}{8}x^4 + \frac{1}{64}x^5 \\
 \hline
 2x^{\frac{5}{2}} - x^{\frac{7}{2}} - \frac{1}{8}x^{\frac{9}{2}} \dots \dots \dots 0 \\
 \hline
 -\frac{1}{8}x^4 + \frac{1}{16}x^5 + \frac{1}{64}x^6 + \frac{1}{256}x^7 \\
 \hline
 2x^{\frac{7}{2}} - x^{\frac{9}{2}} - \frac{1}{4}x^{\frac{11}{2}} - \frac{1}{16}x^{\frac{13}{2}} \dots \dots \dots 0 \\
 \hline
 -\frac{1}{16}x^5 + \frac{1}{64}x^6 - \frac{1}{256}x^7 \\
 \hline
 \dots \dots \dots \text{&c.}
 \end{array}$$

So that $x - x^{\frac{1}{2}}$ is $= x^{\frac{1}{2}} - \frac{1}{2}x^{\frac{3}{2}} - \frac{1}{8}x^{\frac{5}{2}} - \frac{1}{16}x^{\frac{7}{2}} - \frac{1}{128}x^{\frac{9}{2}}$, &c.

And after the same Manner may any common surd Quantity be reduced into an infinite

nite Series. But with much greater Ease and Expedition,

87. All manner of fractional and surd Quantities, may be reduced into infinite Series, by a most curious and excellent Theorem invented for that Purpose, by the Great and Illustrious Sir *Isaac Newton*, called his

UNIVERSAL THEOREM,

$$\text{which is this, } \overline{P+PQ}^{\frac{m}{n}} = P^{\frac{m}{n}} + \frac{m}{n} A Q + \frac{m-n}{2n} B Q + \frac{m-2n}{3n} C Q + \frac{m-3n}{4n} D Q + \frac{m-4n}{5n} E Q + \&c. \text{ wherein it must be ob-}$$

served that, $P+PQ$ represents the Quantity, whose Root, Dimension, or Root of the Dimension, is required to be thrown into an infinite Series; P the first Term of that Quantity, which must be the greatest; Q the rest of the Terms divided by the first; $\frac{m}{n}$ the numerical

Index of the Dimension or Power of $P+PQ$, whether that Power be Affirmative or Negative, Integral or Fractional; and $A, B, C, D, E, \&c.$ the Terms in the Series or Quotient already

found, *i. e.* $A = P^{\frac{m}{n}}$, $B = \frac{m}{n} A Q$, $C = \frac{m-n}{2n} B Q$
 $D =$

$D = \frac{m-2n}{3n} CQ$, $E = \frac{m-3n}{4n} DQ$, or A
 $=$ the first Term, $B =$ the second Term,
 $C =$ the third Term, $D =$ the fourth Term,
 $E =$ the fifth Term, &c. — The following
 Examples will explain, and shew the great
 Use of this curious Theorem.

EXAMPLE I.

88. Let it be required to throw $a^2 + x^2|^{\frac{1}{2}}$ into
 an infinite Series, a^2 being greater than x^2 .

$$\begin{aligned}
 &\text{HERE } P = a^2, Q = \frac{x^2}{a^2}, m = 1, n = 2, A = \\
 &P_n^m = a, B = \frac{m}{n} AQ = \frac{x^2}{2a}, C = \frac{m-n}{2n} BQ = - \\
 &\frac{x^4}{8a^3}, D = \frac{m-2n}{3n} CQ = \frac{x^6}{16a^5}, E = \frac{m-3n}{4n} DQ \\
 &= \frac{5x^8}{128a^7}, \&c. \text{ therefore } a^2 + x^2|^{\frac{1}{2}} = a + \frac{x^2}{2a} \\
 &\frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7}, \&c.
 \end{aligned}$$

Ex -

EXAMPLE II.

89. Required to put $\sqrt{x-x^2}$ into an infinite Series; x being greater than x^2 .

Here $P=x$, $Q=-\frac{x^2}{x}=-x$, $m=1$, $n=2$,
 $A=x^{\frac{1}{2}}$, $B=-\frac{1}{2}x^{\frac{1}{2}}$, $C=-\frac{1}{8}x^{\frac{1}{2}}$, $D=-\frac{1}{16}x^{\frac{1}{2}}$,
 &c. therefore $\sqrt{x-x^2}=x^{\frac{1}{2}}-\frac{1}{2}x^{\frac{1}{2}}-\frac{1}{8}x^{\frac{1}{2}}-\frac{1}{16}x^{\frac{1}{2}}$,
 &c.

EXAMPLE III.

90. Required to put $\frac{b}{a+x}$ into an infinite Series; x being less than a .

$\frac{b}{a+x}$ is $=b \times \frac{1}{a+x} = b \times \overline{a+x}^{-1}$, $P=a$,
 $Q=\frac{x}{a}$, $m=-1$, $n=1$, $A=a^{-1}=\frac{1}{a}$, $B=-$
 $\frac{x}{a^2}$, $C=\frac{x^2}{a^3}$, $D=-\frac{x^3}{a^4}$, $E=\frac{x^4}{a^5}$, &c. there-
 fore $b \times \overline{a+x}^{-1} = b \times \left\{ \frac{1}{a} - \frac{x}{a^2} + \frac{x^2}{a^3} - \frac{x^3}{a^4} + \right.$
 $\left. \frac{x^4}{a^5}, \text{ \&c. that is, } \frac{b}{a+x} = \frac{b}{a} - \frac{bx}{a^2} + \frac{bx^2}{a^3} - \frac{bx^3}{a^4} \right.$
 $\left. + \frac{bx^4}{a^5}, \text{ \&c.} \right.$

E x .

EXAMPLE IV.

91. Required to put $\frac{a^2}{a+x}^2$ into an infinite Series; where a is greater than x .

$$\begin{aligned} \frac{a^2}{a+x}^2 \text{ is } &= a^2 \times \overline{a+x}^{-2}, P=a, Q=\frac{x}{a}, m= \\ &-2, n=1, A=a^{-2}=\frac{1}{a^2}, B=-\frac{2x}{a^3}, C=\frac{3x^2}{a^4} \\ D=-\frac{4x^3}{a^5}, E=\frac{5x^4}{a^6}, \&c. \text{ therefore } a^2 \times \\ \overline{a+x}^{-2} &= a^2 \times \left\{ \frac{1}{a^2} - \frac{2x}{a^3} + \frac{3x^2}{a^4} - \frac{4x^3}{a^5} + \frac{5x^4}{a^6} \right. \\ \&c. \text{ that is, } \frac{a^2}{a+x}^2 &= 1 - \frac{2x}{a} + \frac{3x^2}{a^2} - \frac{4x^3}{a^3} + \\ \frac{5x^4}{a^4}, \&c. \end{aligned}$$

29. But we may find the Series answering to any proposed Quantity, by the following *uni-*

$$\begin{aligned} \text{versal Theorem, viz. } P + P Q^{\frac{m}{n}} &= P^{\frac{m}{n}} \times \left\{ 1 + \right. \\ \frac{m}{n} Q + \frac{m}{n} \times \frac{m-1}{2n} Q^2 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \\ Q^3 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{m-3n}{4n} Q^4 + \&c. \end{aligned}$$

(which indeed is the same as the other, tho' differently expressed) with still more Ease and Expe-

DOCTRINE of FLUXIONS. 121

Expedition; and that without any previous Deduction; if we consider that both the Numerators and Denominators of the Fractions $\frac{m}{n}, \frac{m-n}{2n}, \frac{m-2n}{3n}, \frac{m-3n}{4n}, \&c.$ are Series of Numbers in arithmetical Progreſſion, which have the ſame common Difference n . This will appear by the following Examples,

EXAMPLE I.

93. Let $\sqrt{x-x^2}^{\frac{1}{2}}$ be required to be put into an infinite Series.

Here $P=x$, $Q=-\frac{x^2}{x}=-x$, $m=1$,

$n=2$, therefore $\sqrt{x-x^2}^{\frac{1}{2}}=x^{\frac{1}{2}} \times 1-\frac{1}{2}x +$

$$\frac{1 \times -1}{2 \times 4} x^2 - \frac{1 \times -1 \times -3}{2 \times 4 \times 6} x^3 +$$

$$\frac{1 \times -1 \times -3 \times -5}{2 \times 4 \times 6 \times 8} x^4 - \&c. \text{ i. e. } = x^{\frac{1}{2}} \times :$$

$$1-\frac{1}{2}x - \frac{1}{2.4} x^2 - \frac{3}{2.4.6} x^3 - \frac{3.5}{2.4.6.8} x^4 - \&c.$$

$$\text{or } x^{\frac{1}{2}} - \frac{x^{\frac{3}{2}}}{2} - \frac{x^{\frac{5}{2}}}{2.4} - \frac{3.x^{\frac{7}{2}}}{2.4.6} - \frac{3.5.x^{\frac{9}{2}}}{2.4.6.8} - \&c.$$

EXAMPLE II.

94. Let $\sqrt{a+x}^{\frac{3}{2}}$ be put into an infinite Series.

R

Here

Here $P=a$, $Q=\frac{x}{a}$, $m=3$, $n=5$, therefore $\overline{a+x}^{\frac{1}{3}} = a^{\frac{1}{3}} \times 1 + \frac{3.x}{5.a} + \frac{3.-2.x^2}{5.10.a^2} + \frac{3.-2.-7.x^3}{5.10.15.a^3} + \frac{3.-2.-7.-12.x^4}{5.10.15.20.a^4} + \&c$, i. e. = $a^{\frac{1}{3}} + \frac{3.x}{5.a^{\frac{2}{3}}} - \frac{6.x^2}{5.10.a^{\frac{5}{3}}} + \frac{6.7.x^3}{5.10.15.a^{\frac{8}{3}}} - \frac{6.7.12.x^4}{5.10.15.20.a^{\frac{11}{3}}} + \&c$.

CHAP. IX.

Of finding the Fluent of a given Fluxion.

95. **A**S it is the Business of the direct Method of Fluxions to find the Fluxion of any given variable Quantity or Fluent, or the Velocity with which it flows at any particular Point; So it is the Business of this Chapter or of the inverse Method of Fluxions, to find or determine the variable Quantity or Fluent when that Velocity or Fluxion is given alone: and this may be done by the following Rules.

96. RULE 1. To find the Fluent of a simple Fluxion: Substitute the flowing Term for its Fluxion, and it will give the Fluent requir'd. Thus the Fluent of ax is ax .

97. RULE

97. RULE 2 To find the Fluent of a fluxionary Expression which is compounded of different Simple ones connected together with the Signs $+$ and $-$: Find the Fluent of each simple Expression by Rule 1. which connect together with the Signs of their respective Fluxions, and it will be the Fluent sought. Thus the Fluent of $\dot{x} + \dot{y} - b\dot{z}$ is $x + y - bz$: And the Fluent of $a^2\dot{x} - b^3\dot{y} + \dot{z}$ is $a^2x - b^3y + z$.

98. RULE 3. To find the Fluent of a fluxional Expression which consists of the Products of two or more variable Quantities drawn into their Fluxions *i. e.* which consists of the Fluxion of each variable Quantity multiplied into the other or Product of the others, as $\dot{x}y + x\dot{y}$ or $\dot{x}yz + x\dot{y}z + xy\dot{z}$; Multiply the flowing Quantities together, and their Rectangle is the Fluent sought. Thus the Fluent of $\dot{x}y + x\dot{y}$ is xy , the Fluent of $\dot{x}yz + x\dot{y}z + xy\dot{z}$ is xyz .

99. RULE 4. To find the Fluent of a fluxional Expression which consists of the Fluxion of any variable Quantity drawn into any Power of that Quantity, contain'd any Number of times, as $2x^2\dot{x}$; Strike out the fluxional Letter, increase the Index of the Power of the variable Quantity by 1, and divide the Coefficient by the Index thus increas'd; and it

R 2

will

will be the Fluent sought. Thus the Fluent of

$2x^2\dot{x}$ is $\frac{2}{3}x^3$, the Fluent of $-\frac{\dot{x}}{x^2}$, *i. e.* of $-\frac{\dot{x}}{x^2}$

$x^{-2}\dot{x}$ is $-x^{-2+1}$ divided by $-2+1$, *i. e.* $\frac{-x^{-1}}{-1}$

or x^{-1} or $\frac{1}{x}$, and universally the Fluent of

$\frac{m}{n}x^{\frac{m}{n}-1}\dot{x}$ is $x^{\frac{m}{n}}$ or of $ax^{\frac{m}{n}}\dot{x}$ is $\frac{an}{m+n}x^{\frac{m+n}{n}}$.

$$ax^{\frac{m+n}{n}} \div \frac{m+n}{n} = \frac{a}{\frac{m+n}{n}} x^{\frac{m+n}{n}} = \frac{an}{m+n} x^{\frac{m+n}{n}}.$$

100. RULE 5. To find the Fluent of a

compounded fluxional Expression like $\frac{b}{a+x}\dot{x}$;

Throw the Expression into an infinite Series (by *Chap. 8.*) and find the Fluent of the Series by the foregoing Rules, and it will be the Fluent sought.

Thus to find the Fluent of $\frac{b}{a+x}\dot{x}$,

I throw it into a Series, which (*Art. 90.*) is $\frac{b}{a}\dot{x} - \frac{bx}{a^2}\dot{x} + \frac{bx^2}{a^3}\dot{x} - \frac{bx^3}{a^4}\dot{x} + \frac{bx^4}{a^5}\dot{x}$ &c. and then find the Fluent of this Series, which (*Art.*

100.) is $\frac{b}{a}x - \frac{bx^2}{2a^2} + \frac{bx^3}{3a^3} - \frac{bx^4}{4a^4} + \frac{bx^5}{5a^5}$ &c.

and this is the Fluent of $\frac{b}{a+x}\dot{x}$ required. A-

gain to find the Fluent of $x - x^2|^{\frac{1}{2}}\dot{x}$ the Expression must be thrown into a Series, which (*Art.*

(*Art.* 89) is $x^{\frac{1}{2}}\dot{x} - \frac{1}{2}x^{\frac{3}{2}}\dot{x} - \frac{1}{8}x^{\frac{5}{2}}\dot{x} - \frac{1}{16}x^{\frac{7}{2}}\dot{x}$, &c. and the Fluent of this Series (*Art.* 99.) is $\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{5}x^{\frac{5}{2}} - \frac{1}{28}x^{\frac{7}{2}} - \frac{1}{72}x^{\frac{9}{2}}$, &c. which is the Fluent of $x - x^2$ required.

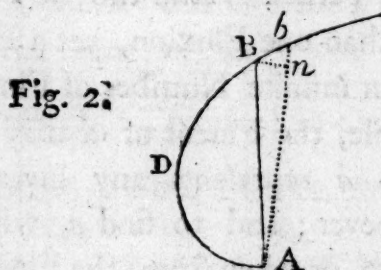
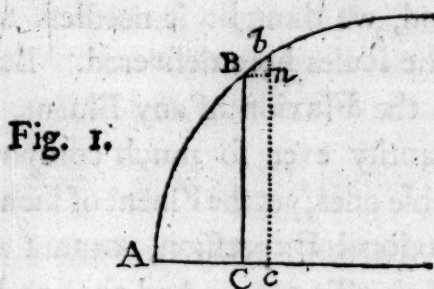
S C H O L I U M.

101. This Chapter being only the Reverse of the Second, we thought it needless to Demonstrate the Rules here delivered. But tho' we can find the Fluxion of any Fluent, be the variable Quantity ever so much compounded with invariable ones; yet the Fluent of such compounded fluxional Expression, cannot always be had in finite Terms. And tho' no Fluent can have more than one Fluxion, yet a Fluxion may have an infinite Number of Fluents; thus, for Example, the Fluent of $\dot{x}$ may be x or $x \pm a$, where a represents any invariable Quantity whatsoever: and to find a , when it must be added to, or taken from, the Fluent x , is call'd correcting the Fluent; but this, in general being very difficult to determine, we shall carefully avoid, in the following Chapters the giving of Examples in which the Fluents of the Fluxions need such Correction.

C H A P. X.

Of finding the Length of a curve Line.

102. **I**N Curves referr'd to an Axis (*Fig. 1.*) let BC be perpendicular to CA , bc indefinitely near parallel to BC , and Bn



equal and parallel to Cc . And in Curves referr'd to a fixt or central Point (*Fig. 2.*) let bA be suppos'd indefinitely near to BA , and the little circular Arch Bn be describ'd with the Radius AB . Put AC (*Fig. 1.*) $=x$, $CB=y$, Curve $AB=z$; $Bn=x'$, $nb=y'$, $Bb=z'$: then
(because

(because Bb may be considered as a little right Line) by 47 E. 1. $Bb = \sqrt{Bn^2 + nb^2}^{\frac{1}{2}}$, i. e. $z' = \sqrt{x'^2 + y'^2}^{\frac{1}{2}}$ or (Art. 3.) $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}^{\frac{1}{2}}$. Put the Ordinate AB (Fig. 2.) $= y$, Curve $AB = z$; $Bn = x'$, $nb = y'$, $Bb = z'$: Then (because Bb may be considered as an indefinitely small right Line, and Bn as a little right Line perpendicular to Ab) as before $z' = \sqrt{x'^2 + y'^2}^{\frac{1}{2}}$ or $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}^{\frac{1}{2}}$ and this is a general Expression for the Fluxion of any curve Line whatsoever. Now by help of the Equation of the given Curve we may find the value of $\dot{x}$, in Terms of y^2 , or of y^2 in Terms of x^2 ; by which $\dot{x}^2$ or $\dot{y}^2$ in this general Expression may be exterminated; and then if we find the Fluent of this we shall have z , or a definitive Expression for the Length of the Curve Line required.

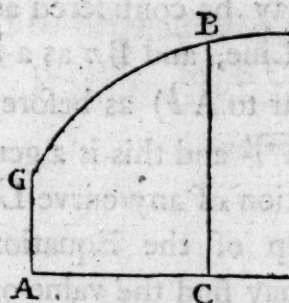
EXAMPLE I.

103. To find the Length of the Curve $GB = z$, whose Equation (putting the given Line $AG = a$, $AC = x$, $CB = y$,) is $2 \times \sqrt{a^2 + x^2}^{\frac{1}{2}} = 3a^2y$.

The Fluxion of this Equation (Art. 13.) is $3 \times \sqrt{a^2 + x^2}^{\frac{1}{2}} \times 2x\dot{x} = 3a^2\dot{y}$, therefore $\dot{y} = \frac{2x\dot{x}}{a^2} \sqrt{a^2 + x^2}^{\frac{1}{2}}$ and $\dot{y}^2 = \frac{4x^2\dot{x}^2}{a^4} \sqrt{a^2 + x^2} =$

$$4a^2x$$

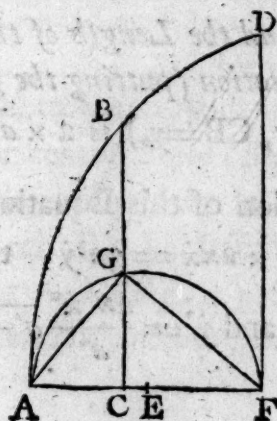
$\frac{4a^2x^2\dot{x}^2+4x^4\dot{x}^2}{a^4}$ which substituted for $\dot{y}^2$ in the general fluxional Expression gives $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}^{\frac{1}{2}}$
 $= \frac{a^2\dot{x}^2 + 4a^2x^2\dot{x}^2 + 4x^4\dot{x}^2}{a^4}^{\frac{1}{2}} = \frac{a^2\dot{x} + 2x^2\dot{x}}{a^2} = \text{the}$



Fluxion of the Curve GB, whose Fluent, by *Art.* 99. is $z = \frac{a^2x + \frac{2}{3}x^3}{a^2}$ or $x + \frac{2x^3}{3a^2} =$ the Length of the Curve GB required.

EXAMPLE II.

104. To find the Length of the common Cyloid.



Put

Put EA or EF, the Radius of the generating Circle $=a$, Absciss AC $=x$, Ordinate CB $=y$, CG $=s$, and AB $=z$, then (as may be found in *Art.* 45.) $\dot{y} = \frac{2a-x}{s} \dot{x}$ and there-

fore $\dot{y}^2 = \frac{(2a-x)^2}{s^2} \dot{x}^2$, which substituted for $\dot{y}^2$ gives the general Expression for the Fluxion of the Curve (*Art.* 102.) $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}^{\frac{1}{2}} = \sqrt{\dot{x}^2 + \frac{(2a-x)^2}{s^2} \dot{x}^2}^{\frac{1}{2}} = \frac{s^2 \dot{x}^2 + 4a^2 \dot{x}^2 - 4ax \dot{x}^2 + x^2 \dot{x}^2}{s^2}^{\frac{1}{2}}$

i. e. (because by the Property of Circles $\overline{GC}^2 = AC \times CF$ or $s^2 = 2ax - x^2$) $\dot{z} =$

$$\frac{4a^2 \dot{x}^2 - 2ax \dot{x}^2}{2ax - x^2}^{\frac{1}{2}} = \frac{2a \dot{x}^2}{x}^{\frac{1}{2}} = 2a^{\frac{1}{2}} \times x^{-\frac{1}{2}} \times \dot{x},$$

and the Fluent of this, by *Art.* 99, is $z = 2a^{\frac{1}{2}} \times 2x^{\frac{1}{2}} = 2 \times \sqrt{2ax}^{\frac{1}{2}} =$ twice the Chord AG (for the Triangles FAG and GAC being Similar, FA : AG :: GA : AC, or $\overline{FA \times AC}^{\frac{1}{2}} = AG$, *i. e.* $\sqrt{2ax}^{\frac{1}{2}} = AG$.) Whence the Length of the Semi-Cycloid AD, is equal to twice the Diameter AF of its generating Circle.

C H A P. XI.

Of finding the Areas of Curve-lined Spaces.

105. **I**N Curves AB referred to an Axis (*Fig. 1.* last Chap.) let bc be conceived indefinitely near and parallel to the perpendicular Ordinate BC , and Bn equal and parallel to Cc ; then, because bn bears no assignable Ratio to BC , bc may be taken as equal to BC or nc , And the Trapezium $BCcb$ as equal to the Parallelogram $BCcn$; But $BCcb$ is the Moment or Increment of the curvilinear Space ABC *i. e.* if we put $AC=x$, $CB=y$ $Cc=x'$, the Moment or Increment of the Space ABC is $=yx'$, and therefore its Fluxion (*Art 3.*) is $=yx$.

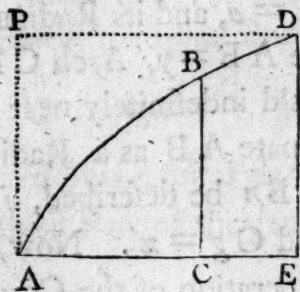
106. In like manner in Spirals or Curves referred to a fixt or central Point, let bA be conceived indefinitely near to BA (*Fig. 2.* last Chap.) and Bn perpendicular to bA ; then, bn having less than any assignable Ratio to nA , $BA n$ may be considered as equal to $BA b$ the Moment or Increment of the curvilinear Space BDA , *i. e.* if we put $AB=y$ and $Bn=x'$; the Moment or Increment of the Space ADB will be $=\frac{1}{2}yx'$; or its Fluxion (that is, the Velocity

locity with which it flows at the Point B) = $\frac{1}{2}y\dot{x}$.

107. Wherefore, when the Curve is referr'd to an Axis, find the Value of y in Terms of x by the help of the Equation of the given Curve, which multiply by $\dot{x}$; then the Fluent of this fluxional Expression being found, will give the Area of the curvilinear Space B A C B requir'd (see Fig. 1. last Chap.) And when the Curve is referred to a central Point A, find the Value of $\dot{x}$ in Terms of y , which may be done by help of the Equation of the given Curve, then multiply this Value of $\dot{x}$ by $\frac{1}{2}y$ and find the Fluent, and it will give the Area of the Space B D A B required (see Fig. 2. last Chap.)

EXAMPLE I.

108. To find the Area of the common or apollonian parabolic Space A D E A; where A E is given = a , and E D = b .



POT

Put $AC = x$, $CB = y$, and the Parameter $= p$; then by the Nature of the Curve $px = y^2$ or $y = \sqrt{px} = p^{\frac{1}{2}}x^{\frac{1}{2}}$, which drawn into x is $yx = p^{\frac{1}{2}}x^{\frac{3}{2}}$, and the Fluent of this (*Art. 99.*) is $\frac{2}{3}p^{\frac{1}{2}}x^{\frac{3}{2}}$ *i. e.* $\frac{2}{3}\sqrt{px^3}$, or (by writing y^2 for px), $\frac{2}{3}x^2y^{\frac{2}{3}}$ or $\frac{2}{3}xy$ which (*Art. 107.*) is = the Area of the indeterminate Space $ABCA$; therefore, by substituting a for x and b for y , we have $\frac{2}{3}ab =$ the Area sought.

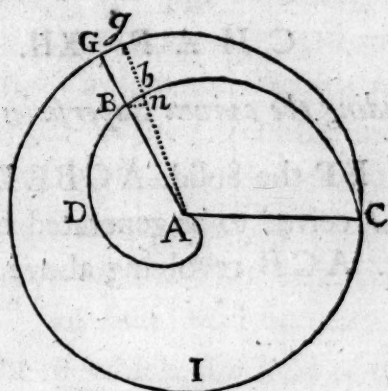
COROLLARY.

The Area of every common or apollonian parabolic Space $ADEA$, is always equal to two-third Parts of its circumscribing Parallelogram $APDE$.

EXAMPLE II.

109. To find the Area of the spiral Space $ADBCA$ of Archimedes.

Put the Circumference of the generating Circle $CIGC = a$, and its Radius AC or $AG = b$; Ordinate $AB = y$, Arch $CIG = z$; let Ag be suppos'd indefinitely near to AG , and with the Ordinate AB as a Radius the little circular Arch Bn be described, *i. e.* let $Bn = x'$, $nb = y'$ and $Gg = z'$. Now, by the Nature of the Generation of the Curve, $b : a :: y$
: z ,



: z , i. e. $z = \frac{ay}{b}$ in Fluxions $\dot{z} = \frac{a\dot{y}}{b}$, but by the similar Sectors or Triangles ABn and AGg , $y : x' :: b : z'$, i. e. $z' = \frac{bx'}{y}$ or $\dot{z} = \frac{bx}{y} \therefore \frac{bx}{y} = \frac{a\dot{y}}{b}$ and $\dot{x} = \frac{ay\dot{y}}{b^2}$ which multiplied by $\frac{1}{2}y$ gives $\frac{1}{2}y\dot{x} = \frac{ay^2\dot{y}}{2b^2}$ whose Fluent (*Art. 99.*) is $\frac{ay^3}{6b^2}$, and (*Art. 107.*) = the Area of the Space $ADBA$, wherefore by substituting b for y , we have $\frac{1}{6}ab^3$ = the spiral Space required.

COROLLARY.

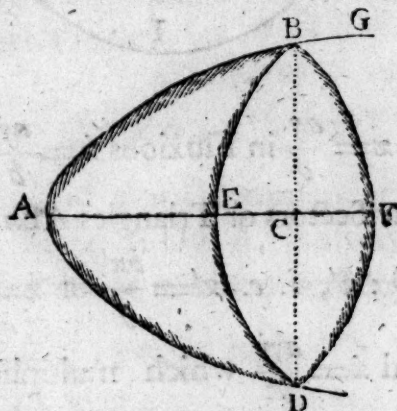
The Area of the spiral Space of *Archimedes*, is = one third Part of the generating Circle.

CHAP.

C H A P. XII.

Of finding the convex Superficies of Solids.

110. **L**ET the Solid ACBEDF be conceived to be generated by the superficial Fig. ACB revolving above AC as an



Axis ; then the Velocity with which its convex Superficies flow, will be equal to the Velocity with which the Curve AG in its Description flowed at the Point B, drawn into the Periphery of the Circle BEDFB described by the Radius CB, that is, the Fluxion of the convex Superficies of the Solid, is equal to the Fluxion of the Curve at the Point B drawn into the Periphery of a Circle whose Radius is CB.

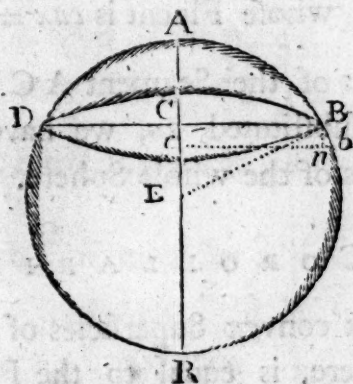
111. Hence

DOCTRINE of FLUXIONS. 135

III. Hence, if we put $AC = x$, $CB = y$, Curve $AB = z$, and $c = 2 \times 3.1416 =$ the Circumference of a Circle whose Radius is 1, then cy will be equal to the Circumference of the Circle described by CB , and (*Art. 102.*) the Fluxion of the Curve $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$; and the general Expression for the Fluxion of the convex Superficies of any Solid will be $= cy\dot{z}$ or $cy \times \sqrt{\dot{x}^2 + \dot{y}^2}$; out of which, by help of the Equation of the given Curve AB , $\dot{x}^2$ or $\dot{y}^2$, &c. may be exterminated, and then the Fluent is found; which will give the Area of the convex Superficies required.

EXAMPLE I.

III2. To find the Superficies of a Sphere, or the convex Superficies of any Segment of it.



Put Radius EA or $EB = a$, $AC = x$, $CB = y$,

$AB = z$: Let $Bn = Cc$ express the very first Moment of the Increase of AC , nb of CB , Bb of AB , *i. e.* let $Bn = x'$, $nb = y'$, and $Bb = z'$, then Bb being considered as a little right Line coinciding with a Tangent to the Point B , the Triangles ECB and $b n B$ will be alike (for $\angle CBn = \angle E B b =$ a right Angle, therefore, $\angle E B n$ being common, $\angle C B E = \angle n B b$; and the Angles at C and n being right, the Angles $C E B$ and $B b n$ must be likewise equal, *ergo*, &c.) Wherefore $EB : BC :: bB : Bn$, *i. e.* $a : y :: z' : x' \therefore z' = \frac{ax'}{y}$ or

(*Art. 3.*) $\dot{z} = \frac{a\dot{x}}{y}$, which substituted for $\dot{z}$, gives the general Expression for the Fluxion of the convex Superficies cyz (*Art. III.*) $= cy \times \frac{a\dot{x}}{y} = ca\dot{x}$, whose Fluent is $cax =$ the convex Superficies of the Segment $ACBD$: And if for x be substituted $2a$, we have $2ca^2 =$ the Superficies of the whole Sphere.

C O R O L L A R I E S.

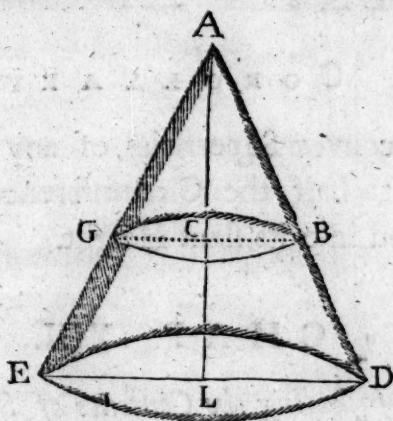
1. THE convex Superficies of any Segment of a Sphere, is equal to the Periphery of a great Circle of that Sphere multiplied into the Altitude of the Segment.

2. THE

2. The whole Surface or Superficies of any Sphere, is equal to the Periphery of its greatest Circle multiplied into its Diameter.

EXAMPLE II.

113. To find the ^{convex}concave Superficies of the right Cone ADE, whose Altitude AL is given = a , and Base diameter DE = b .



Put $AC = x$, $CB = y$. By Sim. Δs $AL : LD :: AC : CB$, i.e. $a : \frac{1}{2}b :: x : y \therefore x = \frac{2ay}{b}$ which put into Fluxions is $\dot{x} = \frac{2a\dot{y}}{b} \therefore \dot{x}^2$

$= \frac{4a^2\dot{y}^2}{b^2}$ which substituted for $\dot{x}^2$ in the gene-

ral Expression for the Fluxion of the convex Superficies, gives $cy \times \sqrt{\dot{x}^2 + \dot{y}^2}^{\frac{1}{2}}$ (Art. III.) =

$cy \times \sqrt{\frac{4a^2\dot{y}^2}{b^2} + \dot{y}^2}^{\frac{1}{2}} = \frac{c}{b} \times \sqrt{4a^2 + b^2}^{\frac{1}{2}} y\dot{y}$, whose

T

Fluent

Fluent is $\frac{c}{2b} \times \overline{4a^2 + b^2}^{\frac{1}{2}} y^2 = \frac{cy^2}{b} \times \overline{a^2 + \frac{1}{4}b^2}^{\frac{1}{2}}$
 $= (\text{because } AD = \overline{AL^2 + LD^2}^{\frac{1}{2}} = \overline{a^2 + \frac{1}{4}b^2}^{\frac{1}{2}})$
 $\frac{cy^2}{b} \times AD = \text{the convex Superficies of the}$
 Cone ABG generated by the plain Figure ACB; and by substituting $\frac{1}{2}b$ for y we have
 $\frac{1}{4}cb \times DA = \text{the convex Superficies of the}$
 Cone ADE.

C O R O L L A R Y.

The convex Superficies of any right Cone, is equal to half the Circumference of its Base multiplied into its slant Height.

C H A P. XIII.

Of finding the Contents of Solids.

114. **L**ET bc be conceived indefinitely near and parallel to the variable Ordinate BC ; and Bn equal and parallel to Cc , the Increment of the variable Absciss AC ; then, because the little Parallelogram $BCcn$ is expressive of the Increment of the plain Fig. ACB (*Art. 105.*) therefore, if a Solid $AEDF$ be conceived to be generated by the Revolution of the curvilinear plane Fig. AED round the Axis AE , the indefinitely little Cylinder generated

which will give the Content of the Solid generated by ACB and then if for x or y we substitute the Value of AE or ED we shall have the Content of the Solid AEDF required, as in the following Examples.

EXAMPLE I.

115. To find the Content of a Sphere or any Segment of it.

PUT $AC = x$, $CB = y$, Diameter $AR = a$, (see Fig. 2. in the last Chap.) then by the Property of Circles $AC \times CR = CB^2$, i. e. $ax - x^2 = y^2$. Now by substituting $ax - x^2$ for y^2 in the general Expression for the Fluxion of the Solid Content we have $cy^2 \dot{x}$ (Art. 114.) $= c\dot{x} \times \overline{ax - x^2} = cax\dot{x} - cx^2\dot{x}$, whose Fluent is $\frac{cax^2}{2} - \frac{cx^3}{3} = \frac{3cax^2 - 2cx^3}{6}$ the Content of the Segment ACBD : and if a be substituted for x we shall have $\frac{3ca^3 - 2ca^3}{6} = \frac{1}{6}ca^3$ the Content of the whole Sphere ABRD. Hence, because four times the Area of a great Circle of the Sphere is $= ca^2$, and the Content of the Cylinder circumscribing the Sphere is $= \frac{1}{4}ca^3$, we have the following

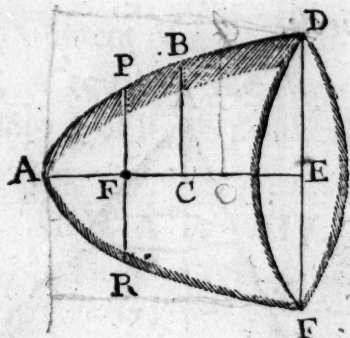
C O R

COROLLARY.

The Content of any Sphere is equal to four Times the Area of its greatest Circle multiplied into $\frac{1}{6}$ th Part of its Axis; or equal to two-third Parts of its circumscribing Cylinder.

EXAMPLE II.

116. To find the Content of the Parabolic Conoid AEDF generated by the Apollonian or common Semi-Parabola AED revolving round the Axis AE; where $AE=b$, $ED=d$.



Put the Parameter PR, which passes thro' the Focus F of the Parabola $=a$; Absciss AC $=x$; and Ordinate CB $=y$; then by the Nature of the Parabola $ax=y^2$. Now by substituting ax for y^2 we shall have the general Expression for the Fluxion of the Solid $cy^2\dot{x}$ (Art. 114.) $=cax\dot{x}$, whose Fluent is $\frac{1}{2}cax^2 =$ (by writing

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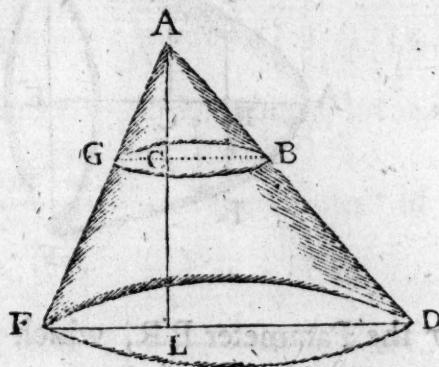
writing y for ax) $\frac{1}{2}cxy^2 =$ the Content of the Parabolic Conoid generated by the Semi-Parabola ACB, and by substituting b for x and d for y , we have $\frac{1}{2}cbd^2 =$ the Content of the Conoid required.

C O R O L L A R Y.

The Content of every common Parabolic Conoid is equal to $\frac{1}{2}$ of its circumscribing Cylinder.

E X A M P L E III.

117. *To find the Content of any Cone whose Base is a Circle.*



PUT the given Altitude $AL = a$, Base-Diameter $DF = b$, let BG be parallel to DL and put $AC = x$, and $c = .7854$ then, by Sim. Δs

$$AL : DF :: AC : BG, \text{ i.e. } a : b :: x : \frac{bx}{a}$$

$= BG$

=BG therefore the Area of the Circle BG = $\frac{bx^2}{a} \times c = \frac{cb^2x^2}{a^2}$ which multiplied by $\dot{x}$ (*Art.*

114.) is $\frac{cb^2x^2\dot{x}}{a^2}$ = the Fluxion of the Content

of the Cone at B, whose Fluent is $\frac{cb^2x^3}{3a^2}$ = the solid Content of the Cone A B G; and by substituting a for x we have $\frac{cb^2a^3}{3a^2} = \frac{1}{3}cb^2a^2$ = the Content of the Cone A D F required.

C O R O L L A R Y.

The solid Content of any Cone whose Base is a Circle, is equal to the Area of its Base multiplied into $\frac{1}{3}$ of its perpendicular Altitude.

C H A P. XIV.

Some Miscellaneous Questions with their Answers.

I.

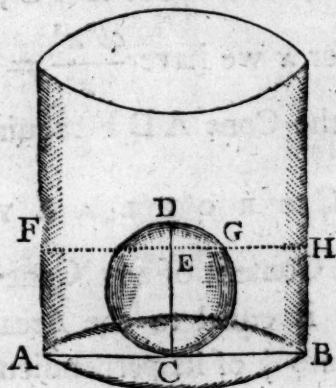
THERE is a cylindrical Tube, whose Diameter is four Inches, in which is contained 18 cubic Inches of Water: Now supposing a heavy Sphere, whose Axis is two Inches, to be thrown into this Tube; 'tis required to find what Part of the said Sphere will
be

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be immersed in the Water, and the Altitude of the Segment above the Surface of it.

S O L U T I O N.

PUT DC the Diameter of the Sphere $= 2 = a$; and EC that Part of it which is $=$ the Altitude of the Water after the Sphere is thrown

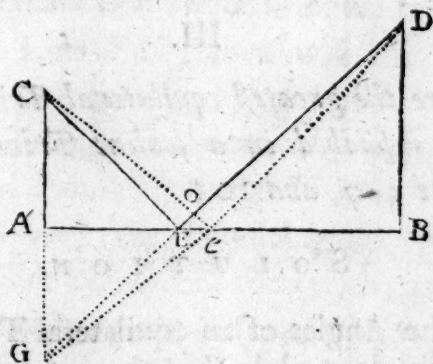


into it $= x$; AB the Diameter of the Tube $= 4 = b$; the Content of the Water in the Tube $= 18 = c$; and $.7854 = f$. Then $ED = a - x$, and by the Property of Circles $DE \times EC = EG^2$, *i. e.* $ax - x^2 =$ the $\square$ of the Radius of the Plane of the Section of the Sphere; therefore the Area of this Plane is $= 4fax - 4fx^2$ which multiplied into x is $4faxx - 4fx^2x =$ the Fluxion of the Segment of the Sphere under the Water, whose Fluent is $2fax^2 - \frac{4}{3}fx^3 + c =$ the Content of the Cylinder $ABHF$
 $= fb^2x$

fb^2x , by Transposition $\frac{4}{3}fx^3 - 2fax^2 + fb^2x = c$ and by dividing both Sides of this Equation by $\frac{4}{3}f$ we have $x^3 - \frac{3}{2}ax^2 + \frac{3}{4}b^2x = \frac{3c}{4f} = 17.18869$. whence x is found $= 1.76 = BE$, and $a - x = 0.24 = ED$ the Altitude of the Segment above the Water. *Q. E. I.*

II.

Given AB, AC , and BD . *Quære the Ratio of the Angles AEC and BED , when $CE + ED$ is a Minimum?*



SOLUTION.

It is evident that the Fluxions of the Lines CE and DE are Negative to each other, and because the Sum of these Fluxions is $= 0$, therefore they are equal to each other, or, because the Increment may be taken for the Fluxion, if e be conceived indefinitely near to E , then

U

 $ce =$

$oe = oE$ and conseq. $\angle oEe = \angle oeE$, *i. e.* the Angles DEB and CEA are in the Ratio of Equality. *Q. E. I.*

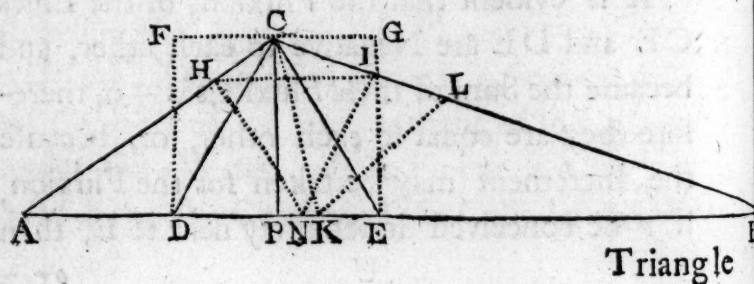
Or, by a plain geometrical Way of Reasoning, thus. — Produce CA to G , making $AG = AC$; then will $GE = EC$, and $\angle AEG = \angle AEC$: But the Sum of GE and ED is the least possible when they are in the same Direction, *i. e.* when $\angle AEG = \angle BED$, as is evident on the least Consideration; therefore, when $CE + ED$ is a *Minimum*, the Angles AEC and BED are equal.

III.

Quære the greatest equilateral Triangle that can be inscribed in a scalene Triangle whose Sides are 4, 7, and 10?

SOLUTION.

As the Angles of an equilateral Triangle are 60° each, it is plain that the extreme Point of either of its Angles cannot fall in with the extreme Point of either of the Angles of the given



Triangle except the largest, viz. the $\angle ACB$. Let CDE then be the greatest, and put its Side $CD=s$, let drop the Perpendicular CP;

$$\text{then by 13 E. 2. } AP = \frac{\overline{AC}^2 + \overline{AB}^2 - \overline{BC}^2}{2AB}$$

$$= 3.35; \text{ and by 47 E. 1. } \overline{AC}^2 - \overline{AP}^2 = \overline{PC}^2 \\ = \overline{CD}^2 - \overline{DP}^2, \text{ i. e. } \overline{AC}^2 - \overline{AP}^2 = s^2 - \frac{1}{4}s^2 \text{ or } \\ 3.7775 = \frac{3}{4}s^2, \text{ whence } s = 2.244. \quad \text{Q. E. I.}$$

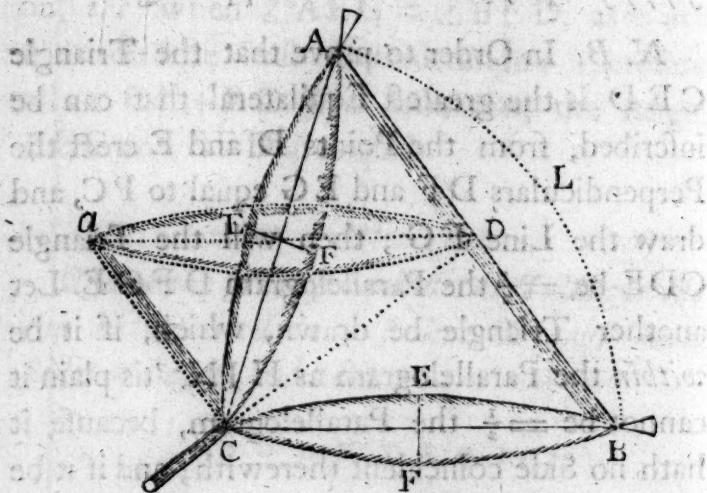
N. B. In Order to prove that the Triangle CED is the greatest Equilateral that can be inscribed, from the Points D and E erect the Perpendiculars DF and EG equal to PC, and draw the Line FG; then will the Triangle CDE be $= \frac{1}{2}$ the Parallelogram DFGE. Let another Triangle be drawn, which, if it be *within* the Parallelogram as HIN, 'tis plain it cannot be $= \frac{1}{2}$ the Parallelogram, because it hath no Side coincident therewith; and if it be *without* and have one Side coincident with one Side of the given Triangle as CLK, neither then can it be $= \frac{1}{2}$ the Parallelogram; for if so, CK would be = CE, and as $\overline{CE}^2 - \overline{EP}^2 = \overline{CN}^2 - \overline{KP}^2$, then subtracting $\overline{CK}^2 = \overline{CE}^2$ from the Equation, $\overline{PK}^2 = \overline{PE}^2$ or $PK = PE$ which is impossible, therefore CDE is the greatest Equilateral that can be inscribed. Q. E. D.

Suppose

IV.

Suppose the Sides of a Pair of Bellows to be two equal Circles; Required the Inclination of Planes when they contain the greatest Quantity of Air.

SOLUTION.

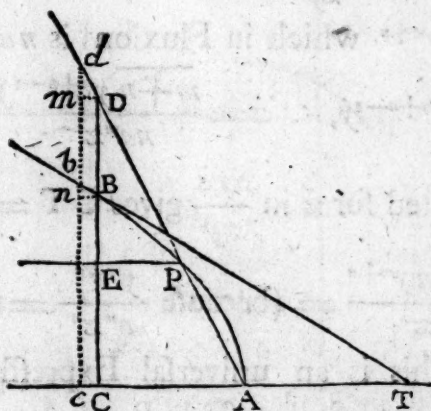


Let AC and BC be the two Sides, and C the Center on which they turn. Now it is plain that if AC be moved up from BC the Point A will describe an Arch of a Circle BLA, and the Chord of that Arch AB will be the Side of the Solid formed by the Bellows so extended: Let us suppose this Solid cut by a plane Perpendicular to AB; then because the Diameter AC is contracted into the Cosine DC while

while the normal Diameter EF continues always in a parallel Situation, and therefore is not at all diminished; it follows that the Plane of that Section will be an Ellipsis: And as the Parts ACD and BCD are equal, if we conceive the upper Part ACD to be moved round into the Situation *a*CD, it will form an oblique Cylinder *a*CB D, and consequently the Content will be greatest when the Altitude is so, *i. e.* when the Sides of the Bellows are normal to each other. *Q. E. I.*

V.

*Quære the Nature of the Curve APB, sup-
posing CE the Distance of the parallel Lines*



AC and PE, as also the Angle CAD, to be given, and $\overline{CE}^m \times \overline{CD}^n = \overline{CB}^{m+n}$?

S O 4

SOLUTION.

Put $CE = a$, $CD = z$, Absciss $AC = x$, Ordinate $CB = y$. Let dc be conceived indefinitely near and parallel to DC ; and Dm , Bn , equal and parallel to Cc ; and put $nb = y'$, $md = z'$; and suppose BT a Tangent to the Curve at the Point B . Then $DC : CA :: dm : md$, i. e. $z : x :: z' : \frac{xz'}{z} = Dm$ or Bn :

And $bn : nB :: BC : CT$, i. e. $y' : \frac{xz'}{z} :: y :$

$\frac{xyz'}{zy} = CT$, or, substituting the Fluxion for the

Increments, $\frac{xyz}{zy} = CT$. But by the Question

$a^m z^n = y^{m+n}$ which in Fluxions is $na^m z^{n-1} \dot{z} = \overline{m+n y^{m+n-1} \dot{y}}$, $\therefore \dot{z} = \frac{m+n y^{m+n-1} \dot{y}}{na^m z^{n-1}}$, and this

substituted for $\dot{z}$ in $\frac{xyz}{zy}$ gives $CT =$

$$\frac{x \cdot \overline{m+n y^{m+n}}}{na^m z^n} = (\text{because } \frac{y^{m+n}}{a^m z^n} = 1,) \frac{m+n}{n} x.$$

Now this is an universal Expression for the Subtangent of all possible Parabolas; therefore when $m = n = 1$, the subtangent will be $2x$, and the Curve the common or Apollonian Parabola. *Q. E. I.*

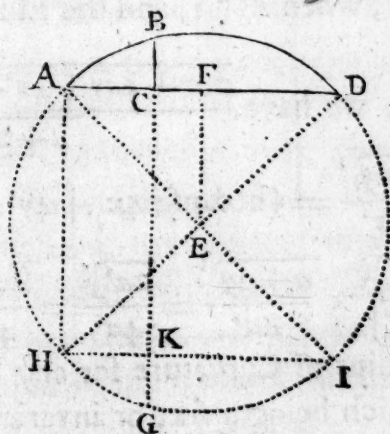
Note,

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Note, Tho' this universal Expression for the Subtangent, seems to differ from that found in *Art.* 39. yet it will be found to express the very same thing, if we consider that the Parameter drawn into x , in the Equation of the Curve there, is equal to $a^{\frac{m}{n}}x$ here, for then, m there, will be equal to $\frac{m+n}{n}$ here.

VI.

In the Curve ABD, whose Property (putting $AC = x$, $CB = y$, and $a = a$ given Line,) is $ax - ay = x^2 + y^2$; required the Radius of Curvature for any Point; and a geometrical Construction to illustrate and confirm the Work.



SOLUTION.

The Fluxion of the given Equation of the Curve,

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Curve, making $\dot{x} = 1$, is $a - ay = 2x + 2yy$, and this fluxional Equation put into Fluxions again, the Fluxion of y being Negative, is $a\dot{y} = 2 + 2y^2 - 2y\dot{y}$. Hence we have $\dot{y} = \frac{a - 2x}{a + 2y}$,

$y^2 = \frac{a^2 - 4ax + 4x^2}{a^2 + 4ay + 4y^2}$, and $\ddot{y} = \frac{2 + 2y^2}{a + 2y}$, i. e. by substituting for $\dot{y}^2$ its equal, $\ddot{y} = \frac{4a^2 + 8ay + 8y^2 - 8ax + 8x^2}{(a + 2y)^3}$ or because by the

Quest. $8x^2 + 8y^2 = 8ax - 8ay$, $\ddot{y} = \frac{4a^2}{(a + 2y)^3}$.

Now by substituting for $\dot{y}^2$ and $\ddot{y}$, these their Values, in the general Expression for the Radius of Curvature, which was found *Art.* 64. = $\frac{1 + \dot{y}^2}{\ddot{y}}$, when $\dot{x} = 1$, and the Fluxion of y Ne-

gative; we have $\frac{2a^2 + 4ay + 4y^2 - 4ax + 4x^2}{(a + 2y)^2}$

$\times \frac{(a + 2y)^3}{4a^2} = (\text{because } 4x^2 + 4y^2 = 4ay - 4ax,)$

$\frac{2a^2}{(a + 2y)^2} \times \frac{(a + 2y)^3}{4a^2} = \frac{2a^2}{4a^2} = \frac{8}{4} = \frac{2}{1} = \frac{a}{2} =$

the Radius of Curvature for any Point required, which being a fixt or invariable Quantity, proves the Curve to be a Circle.

Now, if the Radius of a Circle be $\frac{a}{2}$, 'tis evident

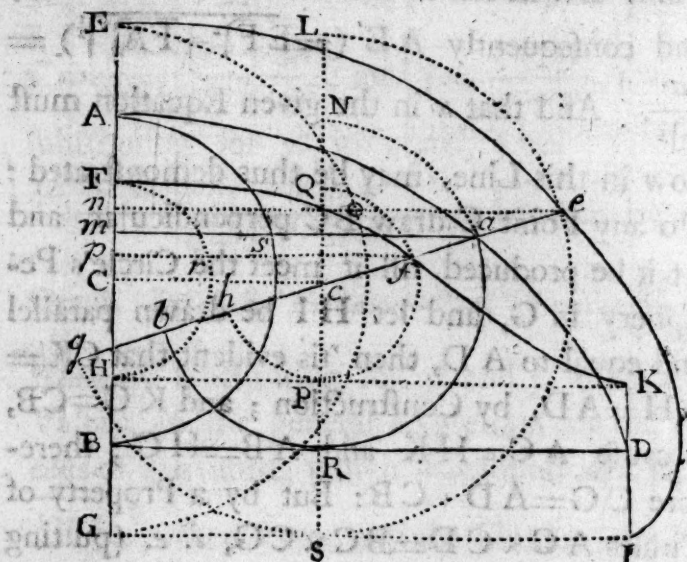
evident that a is the Side of the inscribed Square, or the Chord of 90° as AD : Therefore, if AD be bisected in F , and the Perpendicular FE drawn $= AF = \frac{1}{2}a$; then the Point E will fall in the Center of the Circle, and consequently $AE (= \sqrt{EF^2 + FA^2}) =$

$$\frac{a}{2^{\frac{1}{2}}}$$

And that x in the given Equation must flow in this Line, may be thus demonstrated: To any Point C draw BC perpendicular, and let it be produced till it meet the Circle's Periphery in G , and let HI be drawn parallel and equal to AD , then 'tis evident that $CK = AH = AD$, by Construction; and $KG = CB$, because $AC = HK$ and $AB = HG$; therefore $CG = AD + CB$: But by a Property of Circles $AC \times CD = BC \times CG$, *i. e.* (putting $AD = a$, $AC = x$, $CB = y$,) $x \times a - x = y \times a + y$ or $ax - x^2 = ay + y^2$.

From the Construction here given, it appears that the Question may be infinitely diversified, so as to be adapted to any regular Polygon of an even Number of Sides that can be inscribed in a Circle.——We see here also a Demonstration of the Justness of the fluxional Calculus, as made use of above.

If a Semicircle A B, be rolled along upon a right Line B D, untill it measure out a Line equal to its Circumference; then, the Curve



AD, describ'd by the Point A, being the common Semicycloid; I say the Curve EI, described by the Point E taken *without* the Circle, will be the Contracted or Curtate Semicycloid; and the Curve FK, described by the Point F taken *within* the Circle, will be the Inflected Semicycloid. Quære the Demonstration?

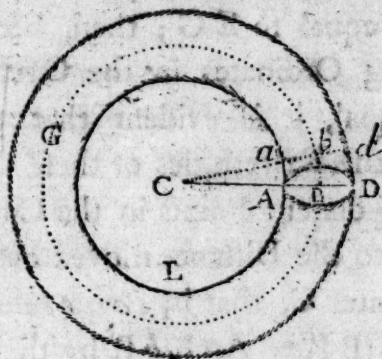
DEMONSTRATION.

As all the Cycloids, properly speaking, are
gene-

generated by the Revolution of one and the same Circle AB ; so therefore their Bases will be all equal to BD , viz. = the Semicircle AB . Let the Generant move on till it come in the Position ab thro' the describing Points a, e, f , draw the Ordinates ma, ne, pf , and draw LS parallel and equal to EG ; then, because the corresponding Ordinates in the Circle are respectively equal, it is evident that the Distances from the Extremities of these Ordinates to the correspondent Points in the Curves will be all equal to the Distance moved over by the common Center C , that is, $sa = oc = rf$, and this also is $= BR = \text{Arch } bR$ by the Generation. Now the Sectors Lce and Scg , or Qcf and Pcb are equal and Similar, and Similar to the equal Sectors Nca and Rcb . Therefore, $cg : cb :: gS : bR$, or (if we put the Semicircle $eg = a$, Semicircle ab or Base $BD = b$,) as $a : b :: \text{Arch } gS : \text{Arch } bR :: Eo : (BR) oe$, i. e. Semicircle $EG : GI :: Eo : oe$, wherefore EeI is the Curtate Semicycloid. And $cb : cb : bP : bR$ or (if we put Semicircle $fb = a$, and Semicircle ab or Base $BD = b$,) as $a : b :: bP : bR :: Fr : (BR) rf$, i. e. Semicircle $FH : HK :: Fr : rf$, wherefore the Curve FfK is the inflected Semicycloid. Q. E. D.

Quare

Quære the Content of the cylindrical Ring $ABDG$; the Radius CA of the inner Circumference being given $=a$; and AD the Diameter of the Ring, or generating Circle, $=b$.



Put any Arch $LA = x$; $.7154 = c$; suppose Cd indefinitely near to CD ; and draw the concentric Circle BG , making $AB = BD = \frac{1}{2}b$: Then may Dd , Bb , Aa , be considered as indefinitely small right Lines; and therefore the Moment of the Ring will be equal to the Area of the Circle ABD drawn into the Increment Bb . Now $CA : Aa :: CB : Bb$ (i. e.) $a : x' :: a + \frac{1}{2}b : x' + \frac{bx'}{2a} = Bb$; and the Area of the Circle $AD = b^2c$; therefore the Moment of the Ring is $= b^2c \times \left(x' + \frac{bx'}{2a} \right) = b^2cx' + \frac{b^3cx'}{2a}$ or its Fluxion $= b^2cx + \frac{b^3cx}{2a}$

$\frac{b^2 cx}{2a}$, and the Fluent of this is $b^2 cx + \frac{b^3 cx}{2a} =$

$1 + \frac{b}{2a} \times b^2 cx =$ the Content of the Ring from

L to A; and by substituting $8ac$ for x , we

have the Content of the whole Ring $= 1 + \frac{b}{2a}$

$\times 8ab^2 c^2 =$ the Area of the generating Circle

A D drawn into the Circumference of the
prickt Circle B G. Q. E. I.

IX.

Suppose the Earth to revolve in a circular Orbit round the Sun as its Center; and the Moon to revolve round the Earth in the same Manner; as also that the Planes of their Orbits do coincide; and that the Diameters of the said Orbits are as 340 to 1; and lastly, that the Moon performs 13.368 Revolutions to every single Revolution of the Earth. Quære the Nature and Description of the Curve generated by the Center of the Moon?

[See the Gentleman's Magazine for September 1743, and the Supplement for that Year.

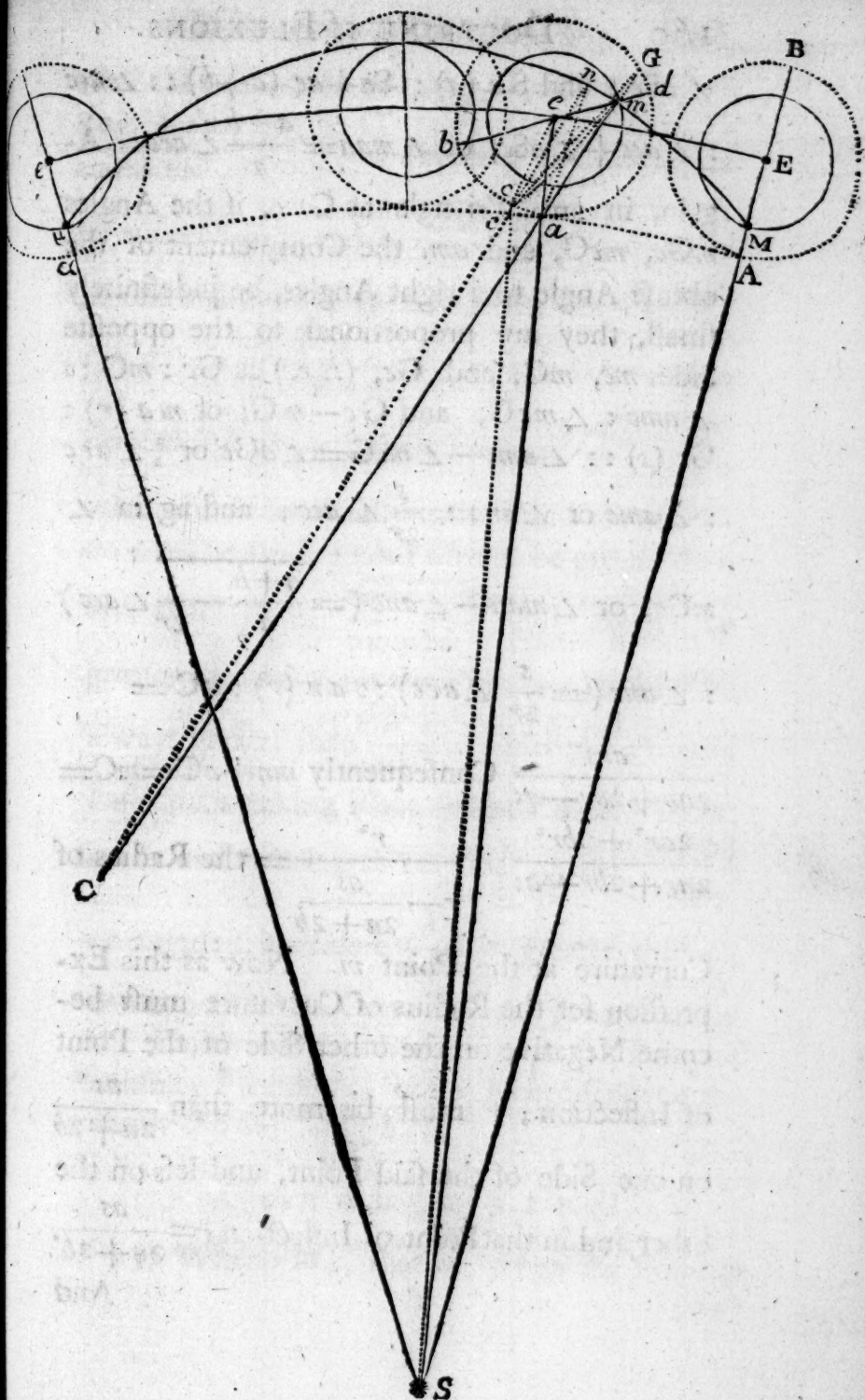
S O L U T I O N.

Let S represent the Sun; E the Earth; Ee,
an Arch of the Orbit of the Earth passed over
by its Center in one Luration of the Moon;
thg

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the Circumference of the Circle EAB = the concentric Arch Aa : Then (because $13.368 - 1 = 12.368$ is = the Number of Lunations in a Year, or one Revolution of the Earth,) when the Moon is in Conjunction with the Sun, the Distance between the Sun and Moon, will be greater than the Distance or Radius SA . Now the Curve described by the Center of the Moon, is the same as that described by a Point M (EM being the Semidiameter of the Moon's Orbit) carried round by the Rotation of the Circle EAB on the Arch Aa ; it is therefore of the cycloidal Kind, having a Point of contrary Flexure, if a Cycloid described by *any* Point within the generating Circle has a Point of Inflection as well upon a circular as upon a rectilinear Base. In Order to determine which, put SA or $Sa = a$, EA or $ea = b$, EM or $em = c$, and $am = r$, and $aG = s$; and let mC be the Radius of Curvature at any Point m , and nC indefinitely near it; and let ac , ac , be the indefinitely small contemporary Arches with mn , and consequently the Triangles amc and anc equal in all respects, and the $\angle man = \angle cac =$ (because the Angles $eaec$ and Sac added to either Side of the Equation makes it 2 right Angles,) $\angle aec + \angle aSc$. Now as $Sa; ea :: \angle aec$
: $\angle aSc$





$\angle aSc$; and $Sc(a) : Sa + ae(a+b) :: \angle aec$

$: \angle aec + \angle aSc$, or $\angle man = \frac{a+b}{a} \angle aec$: A-

gain, in any Triangle as Gmc , if the Angles mGc , mcG , and amc the Complement of the obtuse Angle to 2 right Angles, be indefinitely small, they are proportional to the opposite Sides mc , mG , and Gc , (*i. e.*) as $Gc : mG :: \angle amc : \angle mcG$, and $Gc - mG$, or $ma(r) : Gc(s) :: \angle amc - \angle mcG = \angle aGc$ or $\frac{1}{2} \angle aec$

$: \angle amc$ or $\angle anc = \frac{s}{2r} \angle aec$; and again $\angle$

mCn , or $\angle man - \angle anc (= \frac{a+b}{a} \angle aec - \frac{s}{2r} \angle aec)$

$: \angle anc (= \frac{s}{2r} \angle aec) :: an(r) : aC =$

$\frac{ars}{2ar + 2br - as}$ Consequently $ma + aC = mC =$
 $\frac{2ar^2 + 2br^2}{2ar + 2br - as} = \frac{r^2}{r - \frac{as}{2a+2b}} = \text{the Radius of}$

Curvature at the Point m . Now as this Expression for the Radius of Curvature must become Negative on the other Side of the Point of Inflection; r must be more than $\frac{as}{2a+2b}$ on one Side of the said Point, and less on the other, and in that Point of Inflection $r = \frac{as}{2a+2b}$.

And

And therefore $Gm \times ma (=rs - r^2) =$

$$\frac{2abs^2 + a^2s^2}{2a + 2b} = (\text{by the Property of Circles}) bm$$

$$\times md = b^2 - c^2, \text{ and } s = \frac{2a + 2b\sqrt{a^2 - c^2}}{2ab + b^2}; \text{ Or}$$

$$\text{to find } r \text{ say } 2ar + 2br = as, \text{ then } Gm \times md (=rs - r^2) = \frac{ar^2 + 2br^2}{a} \text{ and (because } Gm \times md =$$

$$bm \times md) = a^2 - c^2, \text{ therefore } r = \sqrt{\frac{ab^2 - ac^2}{a + 2b}},$$

when the Point m falls in the Point of Inflection; but as $ma (r)$ must always be greater than

md , that is, as $\sqrt{\frac{ab^2 - ac^2}{a + 2b}}$ must always be

greater than $b - c$, so therefore $(em)c$ must be always greater than $\frac{b^2}{a + b}$ in order to a Point of

Inflections taking Place in the Curve. Now,

a , b , and c , being as 12.368, 1, and $\frac{13.368}{340}$

or 0.03911, therefore $\frac{b^2}{a + b}$ is greater than c ;

consequently the Curve $Mm\mu$ generated by the Center of the Moon has not a Point of contrary Flexure, or is no where Convex towards the Sun. *Q. E. I.*

COROLLARIES.

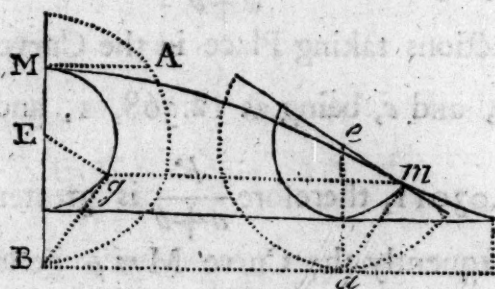
1. When $r = s$, that is, when the Point m
Y coin-

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coincides with d , the Radius of Curvature will be $= \frac{2ar^2 + 2br^2}{ar + 2br}$ or $\frac{a+b}{a+2b}2r$, and $aC = \frac{a}{a+2b}r$, by Analogy $a+2b : a :: r : aC$, that is $SB : SA :: ma : aC$.

2. When a is infinite and $r=s$, that is, when the Base becomes a right Line, and the Point m coincides with d , or the Curve is the common Cycloid; the Radius of Curvature will be $=2r$: For then $2br^2$ and $2br$ will be infinitely little in comparison of $2ar^2$ and ar , and therefore may be rejected.

3. When a is infinite, that is, when the Base degenerates into a right Line, or the Curve is the common Inflected or Interior Cycloid, as in



the annexed Fig. if the Tangent Bg be drawn, the Point of Inflection will be in the Line gm parallel to the Base; that is, if AM be perpendicular to MB , when the Point A comes to the Base, the Point M will be in the Point of

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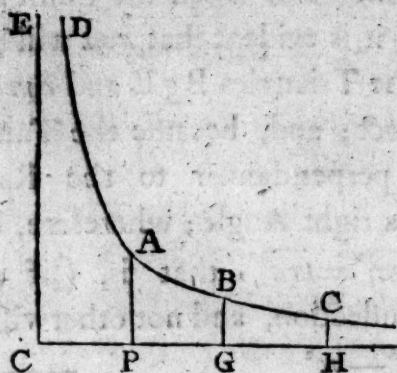
of Inflection. For when the Point m is in the Line gm , it is evident that am will be equal to Bg , and the Triangles BgE and ame , equal in every respect; and, because the Tangent to a Circle is perpendicular to the Radius, the $\angle ame =$ a right Angle; wherefore, by 47E.I. $\overline{ae}^2 - \overline{em}^2 = \overline{ma}^2$, that is, (if m be the Point of Inflection, and not otherwise,) $b^2 - c^2 = r^2 = \frac{ab^2 - ac^2}{a + 2b}$, i. e. $b^2 - c^2 = \overline{b^2 - c^2} \times \frac{a}{a + 2b}$ but $2b$ being infinitely little in respect of a , therefore $\frac{a}{a + 2b}$ is infinitely near to Equality with $\frac{a}{a}$, i. e. $= 1$; ergo, &c.

X.

The Fluxion of the Hyperbolic Logarithm of any Quantity, is equal to the Fluxion of that Quantity divided by the Quantity itself. Quære the Demonstration? (See Art. 14.)

SOLUTION.

Let $DABC$ be an Hyperbola, whose Asymptotes are CE and CH , and its Parameter AP , which by a Property of the Curve is $= PC$. Now if AP or PC be made $= 1$, and GB or HC drawn parallel to PA ; then the
Space



Space PABG will be the hyperbolic Logarithm of CG; and the Space PACH the hyperbolic Logarithm of CH; or the Space GBCH the hyperbolic Logarithm of $\frac{CH}{CG}$:

For by the Property of the Curve when CP, CG, CH, &c. are in geometrical Proportion continued, the Spaces PABG and GBCH, &c. will be equal; that is, the Spaces PABG, PACH, &c. will be in arithmetical Progression. So that the Fluxion of the Space PABG is the Fluxion of the hyperbolic Logarithm of CG: Now the Fluxion of this Space, putting $PG+x$ and GB, perpendicular to GC= y , is $=xy$ by *Art.* 105. but by the Nature of the Curve $CG:CP::PA:GB$, *i. e.* $1+x:1::$

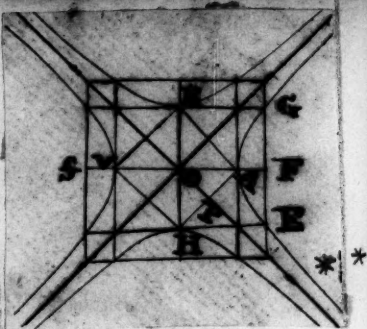
$1:y=\frac{1}{1+x}$ wherefore $\frac{\dot{x}}{1+x} =$ the Fluxion of the hyperbolic Logarithm of $1+x$. But the



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The ordinate which passes through the focus of the Hyperbola, is called the latus rectum, or right parameter. Now the author asserts, that, as the asymptotes are at right angles to each other, OP , or its equal AP , is equal to the parameter. The falsity of this assertion is to be proved.



The Exegesis.

$EG = \text{parameter}$

$EF = \text{semi-parameter} = OH = OA$.

But OA is the hypotenuse of the right-angled triangle OPA , and, therefore, not equal to OP or its equal AP ; consequently OP or AP cannot be equal to the semi-parameter FE , much less equal to the parameter EG . Which was to be proved.

Or thus.

As the parameter is a third proportional to the major and minor axes, we have

$AV : HK :: HK : EG$, but

$AV = HK$, therefore

$HK = EG$. Again

$OH = OA = EF$, the semi-parameter.

But, as above, OA is greater than OP , or PA , and therefore neither of these, OP nor PA can be equal to EF , the semi-parameter, much less equal to EG the parameter. Q.E.D.

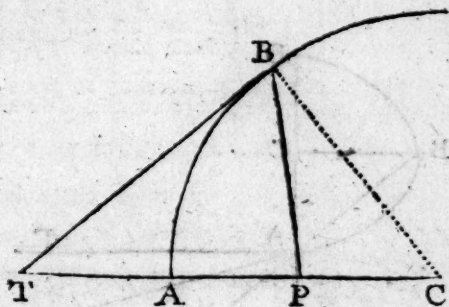
The Author ought to have said, that if $OP = PA$ are given and are at right-angles to each other, the parameter is known and will be, in such a case, $2\sqrt{2 \times PA^2}$. L. E. R.M.A.

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the Fluxion of $1+x$ divided by $1+x$ is likewise $= \frac{\dot{x}}{1+x}$, or the Fluxion of the Space PABG, is equal to the Fluxion of CG divided by CG itself. Q. E. D.

XI.

Given the Point P in the Radius CA of the given Circle AB, &c. Quære the Point B, to which a Tangent TB and right Line PB being drawn, the Angle PBT may be a Minimum or the least possible?



SOLUTION.

Draw the Radius CB. Now, because the Tangent to a Circle is always perpendicular to the Radius, it is evident that the Angle PBT will be the least possible, when the Angle PBC is the greatest: But, because $CB : \text{Sine } \angle BPC :: CP : \text{Sine } \angle PBC$; therefore the Angle PBC will

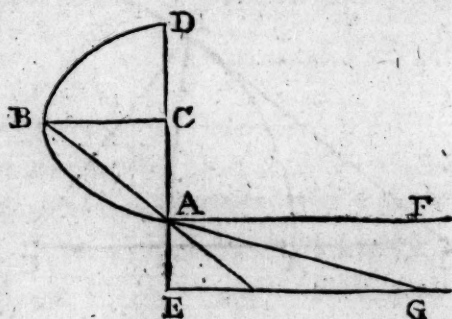
will be greatest, when the Angle CPB is a right one ; Therefore the Angle PBT will be the least possible when PB is perpendicular to CA. Q. E. I.

CHAP. XV.

QUESTIONS.

I. **T**HERE is a Door 6 Feet high, being the Opening of a long Passage against a Street 8 Feet wide. Quære the longest Ladder that can be put in at that Door?

II. Let the Line AG, situated as in the *Fig.*



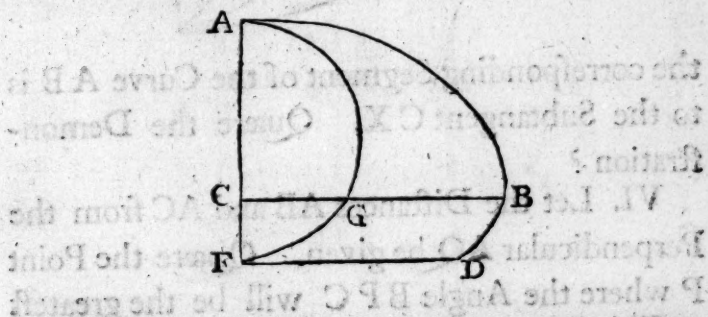
between the Parallels AF and EG, be given $=a$; as also the Distance $AE=b$; then let the End G of the said Line AG, advance towards E, whilst the other End of the said Line is elevated as much as the Point A will give leave, till the said Line comes to stand perpendicular on the Line EG from E to D :

By

By this Motion 'tis plain that the upper End of the Line AG will describe the Curve ABD: Quære the Nature of the said Curve, as also BC its greatest Distance from the Line AD?

III. If a Ball, whose Diameter is 4 Inches, be thrown into a conical Glass, $\frac{1}{2}$ full of Water, whose Diameter is 5 Inches and Altitude 6; how much of the Ball will be immers'd in the Water?

IV. Let ABD be a given curtate Semi-



cycloid, *i. e.* a Semicycloid the Circumference of whose generating Semicircle AGF is greater than its Base FD. Quære the Point C, in the Diameter AF, where the Ordinate CB is a *Maximum* or the greatest possible?

V. Let AB be any given geometrical Curve, and TB a Tangent to it at any Point B: Then if another Curve AD, be so drawn, as that its Ordinate DC, shall always be in a given Ratio to the corresponding Segment of the former Curve AB; I say, that BT will be to TC, as the

Required the fluent of $\frac{x}{a^2} \sqrt{a^4 - b^2 x^2}$?

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$$1. \sqrt{a^4 - b^2 x^2} = a^2 - \frac{b^2 x^2}{2a^2} - \frac{b^4 x^4}{8a^6} - \frac{b^6 x^6}{16a^{10}} - \frac{5b^8 x^8}{128a^{14}}$$

$$2. \text{Mult. by } \frac{x}{a^2} \text{ produces } x - \frac{b^2 x^3}{2a^4} - \frac{b^4 x^5}{8a^8} - \frac{b^6 x^7}{16a^{12}} - \frac{5b^8 x^9}{128a^{16}}$$

$$3. \text{Fluent } x - \frac{b^2 x^3}{2.3a^4} - \frac{b^4 x^5}{5.8a^8} - \frac{b^6 x^7}{7.16a^{12}} - \frac{5b^8 x^9}{9.128a^{16}} \quad \&c$$

see Holladay's Fluxions p. 166.



$$a + \frac{x^2}{2a} - \frac{a^4}{4a^3} + \frac{x^6}{16a^5} - \text{This } + \text{by } 2a =$$

$$2ax + \frac{2x^3}{2a} - \frac{2x^5}{8a^3} + \frac{2x^7}{16a^5} -$$

$$2ax + \frac{x^3}{3a} - \frac{x^5}{20a^3} + \frac{x^7}{56a^5} -$$

Therefore the flux of $2a\sqrt{a^2 + x^2} =$

$$2ax + \frac{13}{3a} - \frac{x^5}{20a^3} + \frac{x^7}{56a^5} \text{ deduc.}$$

$$5000 + 5000 - \frac{20}{20} + \frac{20}{20} + \frac{20}{20} + \frac{20}{20}$$

Handwritten notes and symbols, possibly representing a sequence or a specific mathematical operation.



$$5000 + 5000 - \frac{20}{20} + \frac{20}{20} + \frac{20}{20} + \frac{20}{20}$$

$$5000 + 5000 - \frac{20}{20} + \frac{20}{20} + \frac{20}{20} + \frac{20}{20}$$

